Graduate Preliminary Examination

Algebra I

18.2.2004
3 hours

Problem 1. (a) Let G be a finite nilpotent group. Show that if m divides the order of G, then G has a subgroup of order m.

(b) Give an example of a finite group G such that m divides the order of G but G does not have a subgroup of order m.

Problem 2. Let Σ be the set of Sylow p-subgroups of some finite group, $|\Sigma| \geq 2$ and let $P \in \Sigma$. Clearly P acts on Σ by conjugation.

(a) Find the fix points of P in the set $\Sigma \setminus \{P\}$ if there are any.

(b) Find the length of the orbits of P containing an element of $\Sigma \setminus \{P\}$.

Problem 3. Here $\mathbb{Q}$ is the ring of rational numbers. Let p be the polynomial $X^3 + 9X + 6$ over $\mathbb{Q}$, and let θ be a root of p.

(a) Write θ^3, θ^4 and θ^5 as $\mathbb{Q}$-linear combinations of 1, θ and θ^2.

(b) Is $1 + \theta$ invertible in $\mathbb{Q}[X]/(p)$? If it is, find the inverse; if it is not, explain why.

Problem 4. Let R be a countable integral domain. Prove that R is a principal ideal domain, provided that the following two conditions hold:

- Any two non-zero elements a and b of R have a greatest common divisor, which can be written in the form $ra + sb$ for some r and s in R.
- If $a_1, a_2, \ldots$ are nonzero elements of R such that $a_{n+1} \mid a_n$ for all positive integers n, then there is a positive integer N, such that if $n \geqslant N$, then a_n is a unit times a_N.

Graduate Preliminary Examination

Algebra I

16.2.2005: 3 hours

Problem 1. A group G is called a residually finite group if for any $1 \neq x \in G$ there exists a normal subgroup N such that $x \notin N$ and $|G/N|$ is finite.

Prove that every finitely generated abelian group is residually finite.

Problem 2. Let $GL(n, F_p)$ denote the group of all non-singular $n \times n$ matrices over a field F_p with p elements.

(a) Show that $n \times n$ strictly upper triangular matrices with 1 on the diagonal (unitriangular matrices) in $GL(n, F_p)$ is a Sylow p-subgroup of $GL(n, F_p)$.

(b) Prove that the number of Sylow subgroups in $GL(2, F_p)$ is $p+1$. Exhibit two distinct Sylow p-subgroups of $GL(2, F_p)$.

Problem 3. Let $R = \mathbb{Q}[x]/(x^2 - 1)$.

(a) Find e and f in R, both non-zero such that

$$e^2 = e,\ f^2 = f, ef = 0,\ e + f = 1.$$

(b) Show that $\phi_e : R \to R$ where $\phi_e(r) = re$ is a ring homomorphism.

(c) Find the kernel of the homomorphism ϕ_e

Problem 4. (a) Let S and K be two rings with identity. Let $\phi : S \to K$ be a bijection satisfying $\phi(ab)\phi(1) = \phi(a)\phi(b)$. Show that $\phi(1)$ is an invertible element of K. (Two sided)

(b) Prove or give counter example to the following statement:

If R and S are two rings with identities 1_R and 1_S respectively, then $\phi(1_R) = 1_S$ for every ring homomorphism ϕ from R into S.

METU - Department of Mathematics
Graduate Preliminary Exam

Algebra I

February, 2009

Duration: 180 min.

1. Given a group G, we can construct a chain $G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \xrightarrow{\pi_3} G_4 \xrightarrow{\pi_4} \cdots$, where $G_1 = G$ and $G_{n+1} = \mathrm{Aut}(G_n)$, and $\pi_n(g)(x) = gxg^{-1}$ for all g and x in G_n, for all positive integers n.

a) Show $\pi_n(G_n) \trianglelefteq G_{n+1}$ for all positive integers n.

b) Assuming $C(G) = \langle 1 \rangle$, show that π_n is injective, and $C_{G_{n+1}}(\pi_n(G_n)) = \langle 1 \rangle$ for all positive integers n.

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2. Let G be a finite group and p be the smallest prime divisor of $|G|$. Prove that if H is a subgroup of index p in G, then $H \trianglelefteq G$.

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3. Let Ω be a set, and for each i in Ω, let K_i be a field. Then let R be the ring $\prod_{i\in\Omega} K_i$. Let $\mathfrak{m}$ be a maximal ideal of R. If x is an element $(x_i : i \in I)$ of R, define $S(x) = \{i \in \Omega : x_i \neq 0\}$.

a) Show that, for all x and y in R, if Ω is the *disjoint* union of $S(x)$ and $S(y)$, then exactly one of x and y is in $\mathfrak{m}$.

b) Show that the homomorphism $x \mapsto x/1$ from R to the localization $R_{\mathfrak{m}}$ is surjective.

c) Find the kernel of the homomorphism in (b).

d) What kind of ring is $R_{\mathfrak{m}}$?

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4. Let R be a ring and e be an idempotent in R, that is, $e^2 = e \neq 0$.

a) Show that eRe is a subring of R and e is the identity of eRe

b) Show that if R is finite and contains no nonzero nilpotent elements, then we have $eRe = eR$.

GRADUATE PRELIMINARY EXAMINATION

ALGEBRA I
Spring 2010

1. Prove that a finite group G is nilpotent if and only if $ab = ba$ whenever a and b are elements of G with $(|a|, |b|) = 1$.

2. Let p and q be distinct primes . Show that there is no simple group of order p^3q.
(Hint: The case $p^3q = 24$ can be treated separately)

3. Let R be a commutative ring with identity 1.

(a) Let $P_1 \subseteq P_2 \subseteq \cdots$ and $Q_1 \subseteq Q_2 \subseteq \cdots$ be chains of prime ideals in R. Show that $\bigcup P_i$ and $\bigcap Q_i$ are also prime ideals of R.

(b) Assume that P and Q are prime ideals in R such that $P \subsetneq Q$. Show that there exist prime ideals P^* and Q^* in R such that $P \subseteq P^* \subsetneq Q^* \subseteq Q$ and there is no prime ideal properly lying between P^* and Q^*. (Hint: You may need to use Zorn's lemma.)

4. Let K be a field. A discrete valuation on K is a function $\nu : K^* \longrightarrow \mathbb{Z}$ satisfying

(i) $\nu(ab) = \nu(a) + \nu(b)$

(ii) $\nu(x + y) \geq min\{\nu(x), \nu(y)\}$ for all x, y in K^* with $x + y \neq 0$.

The set $R = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$ is called a valuation ring of ν.

(a) Prove that R is a subring of K which contains the identity.

(b) Prove that for each non-zero element $x \in K$ either x or x^{-1} is in R.

(c) Prove that an element x is a unit of R if and only if $\nu(x) = 0$.

(d) Give an example of a discrete valuation on the field of rational numbers.

M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

ALGEBRA I

1. Let G be a finite group and p be a prime number dividing $|G|$, and let $X = \{x \in G \mid x^p = 1\}$. Prove that $|X|$ is divisible by p.

   (Hint: Let $P \in Syl_p G$. Consider the action of P on X by conjugation and verify that $X \cap C_G(P)$ is a subgroup of P .)

2. Let G be a finite group.

   a) Prove that $N_G(N_G(P)) = N_G(P)$ for each Sylow p-subgroup of G for a prime number p dividing $|G|$.

   b) Prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) for the following:

   (i) G is nilpotent.

   (ii) H is properly contained in $N_G(H)$ for each proper subgroup H of G.

   (iii) Every Sylow p-subgroup of G is normal in G for a prime number p dividing $|G|$.

   (iv) G is the direct product of its Sylow subgroups.

3. Let R be a commutative ring with 1, and let $\mathcal{S}$ be the set of all ideals of R in which every element is a zero divisor. Assume that $\mathcal{S} \neq \phi$.

   (a) Prove that $\mathcal{S}$ has maximal elements with respect to inclusion.

   (b) Prove that every maximal element of $\mathcal{S}$ with respect to inclusion is a prime ideal.

(c) Let D be the set of all zero divisors of R and P be a prime ideal of R. Prove that $P \cap D$ is a prime ideal of R and D is a union of prime ideals.

(d) Give an example of a commutative ring R with 1 such that D is not an ideal and describe the decomposition of D in your example as a union of prime ideals.

4. Let R be a commutative ring with 1, and let X be the set of prime ideals of A. For each subset S of R, let $V(S)$ denote the set of all prime ideals of R which contain S. Prove that

(a) If I is the ideal generated by S, then $V(S) = V(I)$.

(b) $V(0) = X$ and $V(1) = \phi$.

(c) If $(S_i)_{i \in I}$ is a family of subsets of R, then V $(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V(S_i)$.

(d) $V(P \cap Q) = V(PQ) = V(P) \cup V(Q)$ for any ideals P, Q of R.

(e) Suppose that R is Noetherian and let I be an ideal of R. Show that $V(I) = X$ if and only if $I^n = 0$ for some positive integer n.

GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, FEBRUARY 2013

FEBRUARY 13, 2013

1.a. Let p, q, r be distinct primes. Show that any group G of order $|G| = p^2q^2r^2$ is abelian if and only if it is nilponent.

1.b. List all the nilpotent groups of order $p^2q^2r^2$, up to isomorphism.

1.c. Give an example of a nilpotent group of order $2^3 3^2 5^2$, which is not abelian.

2.a. Let G be a finite group of order 105 and let n_p denote the number of Sylow p-subgroups of G, where $p \in \{3, 5, 7\}$. Show that we cannot have simultaneously $n_p > 1$, for all p. Conclude that G is not simple.

2.b. Show that any group G of order 105 has indeed a unique Sylow-7 subgroup. (*Hint: If G does have not a normal subgroup of order 7, then show that it has a normal subgroup of order 15. In this case, a Sylow 7-subgroup acts on this subgroup of order 15, by conjugation. Next show that G is abelian, which yields a contradiction.*)

2.c. Is there any nonabelian group G of order 105? Explain your answer.

3.a. Let R be a commutative ring with unity 1. If $I \subseteq R$ is an ideal then its radical is defined to be subset

$$\sqrt{I} \doteq \{x \in R \mid x^n \in I, \text{ for some } n \in \mathbb{N}\}.$$

Show that $\sqrt{I}$ is an ideal of R.

3.b. Prove that for any prime ideal $P \subseteq R$ its radical is equal to itself: $P = \sqrt{P}$.

4.a. Let $f : R \to S$ be a surjective ring homomorphism, where R is a PID. Show that S is an integral domain if and only if S is a field.

4.b. Let F be any field. Show that any ring homomorphism $f : F[x] \to \mathbb{Z}$ is trivial (i.e., it is the zero homomorphism).

4.c. Construct infinitely many distinct ring homomorphisms from $\mathbb{Q}[x]$ to $\mathbb{Q}$.

METU Mathematics Department
Graduate Preliminary Examination
Algebra I, January 2014

1. Prove or disprove the following claims:

   - No group can have exactly two subgroups of index two.
   - The abelian group $\mathbb{Q}/\mathbb{Z}$ is finitely generated.

2. Let H be a proper subgroup of a finite group G. Show that

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

3. Let R be a principal ideal domain and let $\mathfrak{p} \subset R$ be a non-zero prime ideal. Show that the localization $R_{\mathfrak{p}}$ is a principal ideal domain and has a unique irreducible element, up to associates.

4. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let $\mathfrak{a} = (3, 1 + \sqrt{-5})$.

   - Show that $\mathfrak{a}$ is not a principal ideal.
   - Show that $\mathfrak{a}^2 = \mathfrak{a}\mathfrak{a}$ is a principal ideal and find a generator.

METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, FEBRUARY 2015

FEBRUARY 9, 2015

1.a) Let G denote the free product of the group with two elements with itself:

$$G = \mathbb{Z}_2 * \mathbb{Z}_2 =< a, b \mid a^2,\ b^2 > .$$

Show that the subgroup generated by the element ab ,

$$N =< ab > \lhd G$$

is an infinite cyclic group and it is normal with index two.

b) Let $H =< a > \leqslant G$. Describe the action of H on N by conjugation. Show that $G = NH$ with $N \cap H = \{1\}$.

c) Conclude that G is isomorphic to the semidirect product $N \rtimes H$ with the action described in part (b).

2.a) Show that any finite group of order 65 is cyclic.

b) Show that any finite group of order 130 has a unique subgroup of order 13.

c) Show that any finite group of order 130 has a subgroup of order 65.

d) Show that up to isomorphism there are only two groups of order 130. Describe them.

3.a) For any integer $n > 1$ consider the ring $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$. Show that any ideal $I \subseteq \mathbb{Z}_n$ is principal. (Note that the ring $\mathbb{Z}_n$ is not necessarily a PID.)

b) Let R be any ring so that there is an **onto** ring homomorphism $f : \mathbb{Z}_n \to R$. Use part (a) to show that R is isomorphic to some $\mathbb{Z}_m$, where $m|n$.

c) Let $g : \mathbb{Z}_n \to \mathbb{Z}_m$ be an injective ring homomorphism and $g(1) = k \in \mathbb{Z}_m$ (we abuse the notation and let k denote its residue class in $\mathbb{Z}_m$). Show that k is an element of order n in the additive group $\mathbb{Z}_m$, $m \mid nk$ and $m \mid k(k-1)$.

d) Let m, n, k be integers with $m, n > 1$, $0 < k < m$ so that $m \mid nk$, $m \mid k(k-1)$ and $k \in \mathbb{Z}_m$ has order n. Show that the map $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ defined by $\phi(r) = kr$, $\forall r \in \mathbb{Z}_n$, is a well defined injective ring homomorphism.

e) List all subrings of $\mathbb{Z}_{30}$. Find a nontrivial ring homomorphism $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{30}$. Is there any other?

4.a) Let $N : \mathbb{Z}[\sqrt{5}] \to \mathbb{Z}$ be defined by the rule

$$N(a + b\sqrt{5}) = (a + b\sqrt{5})(a - b\sqrt{5}) = a^2 - 5b^2.$$

Show that $N((a + b\sqrt{5})(c + d\sqrt{5})) = N(a + b\sqrt{5})\ N(c + d\sqrt{5})$, for all $(a + b\sqrt{5})$ and $(c + d\sqrt{5}) \in \mathbb{Z}[\sqrt{5}]$.

b) Use part (a) to show that if $|N(a + b\sqrt{5})| = p$ is a prime integer then $a + b\sqrt{5}$ is an irreducible element in $\mathbb{Z}[\sqrt{5}]$.

Graduate Preliminary Examination

Algebra I

22.9.2004; 3 hours

Problem 1. Let p be a prime number. If G is an infinite p-group such that every proper non-trivial subgroup of G has order p, prove that:

(a) $p > 2$;

(b) G must be a simple group.

(Recall that a group G is called simple if it has no non-trivial normal subgroup.)

Problem 2. Let $\mathbb{Q}$ be the additive group of rationals.

(a) Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

(b) Prove that if G is a group with center $Z(G)$ such that $G/Z(G)$ is isomorphic to a subgroup of $\mathbb{Q}$, then G is abelian.

(c) Prove that no non-trivial subgroup of $\mathbb{Q}$ can be isomorphic to the full group of automorphisms of a group.

Problem 3. Let R be a quadratic integer $\mathbb{Z}[\sqrt{-5}]$. Let $I = (2, 1 + \sqrt{-5})$ be an ideal of R.

(a) Is I a principal ideal in $\mathbb{Z}[\sqrt{-5}\ ]$? Justify your answer.

(b) Is $I^2 = II$ a principal ideal in $\mathbb{Z}[\sqrt{-5}\ ]$ Justify your answer.

Problem 4. Let R be a commutative ring with identity. If I is an ideal of R , then $\sqrt{I} = \{r \in R \ : \ r^n \in I \quad \text{for some positive integer } n\}$. A proper ideal I is called a primary ideal if whenever $ab \in I$ we have either $a \in I$ or $b \in \sqrt{I}$.

(a) Prove that if I is a primary ideal, then $\sqrt{I}$ is a prime ideal.

(b) Is $(4, x)$ a prime ideal in $\mathbb{Z}[x]$? Explain.

(c) Is $(4, x)$ a primary ideal in $\mathbb{Z}[x]$? Explain.

PRELIMINARY EXAMINATION
ALGEBRA I
FalI 2005
September 14^{th}, 2005

Duration: 3 hours

1. Determine all groups with exactly three distinct subgroups.

2. Let A be an abelian group denoted additively. Let ϕ be an endomorphism of A. Show that if ϕ is nilpotent, then $1+\phi$ is an automorphism of A.

Hint: Consider the factorization of $1+\phi^n$ (with n odd) in the ring End A. Note that 1 means the identity map of A.

3. A ring R is called radical if for every $x \in R$, there exists $y \in R$ such that $x+y+xy=0$.

a) Let R be a ring. If every element of R is nilpotent, then show that R is radical.

b) Show that $R = \left\{ \frac{2x}{2y+1} | x, y \in \mathbb{Z} \text{ such that } (2x, 2y+1) = 1 \right\}$ is a radical ring.

c) Prove or disprove: In a radical ring every element is nilpotent.

4. Let R be a commutative ring with identity 1. A subset S of R is called a multiplicative set if it is closed under multiplication, contains 1, and does not contain the zero element.

a) Prove that an ideal I of R is prime if and only if there is a multiplicative set S such that I is maximal among ideals disjoint from S.

b) Prove that the set of all nilpotent elements of R equals the intersection of all the prime ideals of R.

Hint: If s is not nilpotent, then $\{1, s, s^2, \cdots\}$ is a multiplicative set.

TMS

Fall 2010

Algebra I

1. Let G be a finite group and suppose $H \leq G$ satisfies the condition that $C_G(x) \leq H$ for all $x \in H - \{1\}$. Show that $gcd(|H|, |G : H|) = 1$.

   Hint: Choose $P \in Syl_p(H)$ and show that $P \in Syl_p(G)$

2. In this question, G stands for a finite group where $C_G(a)$ is abelian for every $a \in G \setminus \{1\}$.

   (a) Give two examples of such a group; one non-abelian nilpotent, one non-nilpotent solvable.

   (b) Assume $Z(G) = 1$. Show that commuting is an equivalence relation on $G \setminus \{1\}$. What are the equivalence classes?

   (c) Let $Z(G) = 1$ and A be a maximal abelian subgroup of G. Show that $A = C_G(a)$ for every $a \in A$ and $\gcd(|A|, [G : A]) = 1$.

3. Let R be a commutative ring.

   Prove that the following are equivalent for R.

   (1) There is a proper ideal P in R such that $P \supseteq I$, for every proper ideal I of R.

   (2) The set of nonunits of R forms an ideal.

   (3) There exists a maximal ideal M of R such that $1 + x$ is a unit, for all $x \in M$.

4. Let R be ring with unity and define

$$N(R) = \{a \in R : a^n = 0 \quad \text{for some} \quad n \geq 1\}$$

   $J(R) = \cap M$, intersection of all maximal ideals in R.

   (a) Show that if R is commutative, then $N(R)$ is an ideal and that

$$N(R) = \cap P, \quad \text{intersection of all prime ideals in R.}$$

   (b) Give an eample to show that if R is not commutative, then $N(R)$ need not be an ideal.

   (c) Give examples R, S of commutative rings such that

$$0 \neq N(R) = J(R)$$

$$0 \neq N(S) \neq J(S)$$

M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

ALGEBRA I

Duration : 180 min.

Each question is 25 pt.

1. Let G be a finite group.

   a) Suppose that M and N are normal subgroups of G such that both G/M and G/N are solvable. Prove that $G/M \cap N$ is solvable.

   b) Prove that G has a subgroup L which is the unique smallest subgroup with the properties of being normal with solvable quotient.

   c) Suppose that G has a subgroup H isomorphic to A_5. Show that $H \subseteq L$.

2. Let G be a group.

   a) Prove that if N is a normal subgroup of G then $G/C_G(N)$ is isomorphic to a subgroup of the group $Aut(N)$ of all automorphisms of N.

   Prove the following assuming that G is a finite group with

   $$\gcd(|G|, |\mathrm{Aut}(G)|) = 1.$$

   b) G is abelian.

   c) Every Sylow subgroup of G is cyclic of prime order.

   d) G is a cyclic group of squarefree order such that if p and q are prime divisors of $|G|$, then we have $p \not\equiv 1(\mathrm{mod} q)$ and $q \not\equiv 1(\mathrm{mod} p)$.

3. Let $\mathbb{F}_2$ be the field with 2 elements and $R = \mathbb{F}_2[X, 1/X]$ for an indeterminate X.

   Prove the following:

   a) The unit group of R is generated by X.

   b) There are infinitely many distinct ring endomorphisms of R.

   c) The ring automorphism group $\mathrm{Aut}(R)$ is of order 2.

4. Let $(R, +, \cdot)$ be a commutative ring with identity $1 \neq 0$, D be a multiplicative set in R $(1 \in D)$, $D^{-1}T = \{\frac{t}{d} \,|\, t \in T,\ d \in D\}$, $S = D^{-1}R$ be the ring of fractions of R with respect to D and $\pi : R \longrightarrow S$ be the ring homomorphism given by $\pi(r) = \frac{r}{1}$.

   Consider the following property for **proper** ideals I of R:
   (*) if $xr \in I$ for some $r \in D$ then $x \in I$.

   a) Show that if Q satisfies (*) and $D = R \backslash Q$ then

   $$\ker(\pi) \subseteq Q \cap \text{Zerodivisors(R)}.$$

   b) Show that if Q satisfies (*) and $D^{-1}Q = D^{-1}J$ then $J \subseteq Q$.

   c) Show that if Q satisfies (*) and $D^{-1}Q$ is a prime ideal then $D \cap Q = \emptyset$ and Q is a prime ideal.

   d) Show that if P is a prime ideal and $D \cap P = \emptyset$ then P satisfies (*) and $D^{-1}P$ is a prime ideal.

METU MATHEMATICS DEPARTMENT
ALGEBRA I
SEPTEMBER 2012 - TMS EXAM

1. Prove that every group of order $3^2 \cdot 5 \cdot 17$ is abelian

2. a) Prove that Aut $S_3 \cong S_3$.

b) Prove that every nonabelian simple group G is isomorphic to a subgroup of Aut$G$.

3. Let R be an integral domain with 1. A non-zero, non-unit element $s \in R$ is said to be **special** if, for every element $a \in R$, there exist $r, q \in R$ with $a = qs + r$ and such that r is either 0 or a unit of R.

Prove the following:

a) If $s \in R$ is a special element, then the principal ideal (s) generated by s is maximal in R.

b) Every polynomial in $\mathbb{Q}[x]$ of degree 1 is special in $\mathbb{Q}[x]$.

c) There are no special elements in $\mathbb{Z}[x]$. (Hint: Apply the definition with $a = 2$ and $a = x$)

4. Let $A_1, \cdots, A_n$ be ideals of the commutative ring R, and let $D = \bigcap_{i=1}^{n} A_i$. Recall that the radical $\sqrt{I}$ of an ideal I of R is defined as $\sqrt{I} = \{x \in R \,|\, x^k \in I \;\text{ for some positive integer } k\ \}$.

(a) Prove that $I \subseteq \sqrt{I}$, $\sqrt{I} = \sqrt{\sqrt{I}}$ and $\sqrt{I} \subseteq \sqrt{J}$ whenever $I \subseteq J$ for ideals I and J in R.

(b) Prove that $\sqrt{D} = \bigcap_{i=1}^{n} \sqrt{A_i}$.

(c) Suppose that D is a primary ideal and D is not the intersection of elements in any proper subset of $\{A_1, \ldots, A_n\}$. Show that $\sqrt{A_i} = \sqrt{D}$ for each $i = 1, \ldots, n$.

(Recall that an ideal $I (\neq R)$ of R is primary if for any $x, y \in R$, $xy \in I$ and $x \notin I \Rightarrow y^{\ell} \in I$ for some positive integer ℓ)

METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, SEPTEMBER 2013

SEPTEMBER 19, 2013

1.a) Let $G = H_1 \cup H_2 \cup H_3$ be a finite group, where each H_i is a proper subgroup of G. Show that $H_i \neq H_j$ if $i \neq j$.

1.b) Show that each H_i has index two in G.

2.a) Show that any group of order $175 = 5^2\,7$ has a unique Sylow 5 and a unique Sylow 7 subgroup. Use this to show that any group of order $175 = 5^2\,7$ is abelian. List all groups of order 175 up to isomorphism.

2.b) Show that the automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the symmetric group on three letters S_3.

2.c) Let G be a finite group and $N \lhd G$ a normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\gcd(6, |G|/4) = 1$. Use Part (b) to show that $N \subseteq Z(G)$; in other words, N is contained in the center of G.

3.a) Let $\mathbb{F}$ be a field with characteristic different from seven. Show that the polynomial $f(x, y) = 7x - x^2y + 2xy^3 - 5x^3y^4 + y^{100} \in \mathbb{F}[x, y]$ is irreducible. Is the same polynomial irreducible as an element of $\mathbb{Z}[x, y]$? Explain.

3.b) State the Eisenstein Criterion for UFD's and use it to show that the polynomial

$$g(x, y, z) = z^{1000} + x + \sum_{i=1}^{999} x^i y^i z^i$$

is irreducible in $\mathbb{C}[x, y, z]$.

4) Let p be a prime integer and $S = \mathbb{Z} - (p)$; the set of all integers not divisible by p. Show that S is a multiplicative subset of $\mathbb{Z}$ and consider the localization

$$\mathbb{Z}_{(p)} \doteq S^{-1}\mathbb{Z} = \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in S\}$$

as a subring of the field of rational numbers. Show that $\mathbb{Z}_{(p)}$ has a unique maximal ideal $m \subset \mathbb{Z}_{(p)}$, consisting of all non units in $\mathbb{Z}_{(p)}$. What is the quotient field $\mathbb{Z}_{(p)}/m$? Prove your answer.

METU Mathematics Department
Graduate Preliminary Examination
Algebra I, Fall 2014

1. Let A be an abelian p-group of exponent p^m. Suppose that B is a subgroup of A of order p^m and both B and A/B are cyclic. Show that there is a subgroup C of A such that $A \cong B \oplus C$ and $B \cap C = \{0\}$.

2. Let $p > q$ be primes.

   (a) Prove that a group of order pq is not simple.

   (b) Show that there is exactly one group of order pq if $p-1$ is not divisible by q.

   (c) Construct a nonabelian group of order pq if $p-1$ is divisible by q.

3. Let R be a commutative ring with identity and let G be a finite group.

   (a) Show that the augmentation map from the group ring $R[G]$ to R given by the formula $f(\sum c_g g) = \sum c_g$ is a ring homomorphism.

   (b) Show that the augmentation ideal, i.e. the kernel of the augmentation homomorphism, is generated by $\{g-1 | g \in G\}$.

   (c) If G is cyclic with generator g_0, then show that the augmentation ideal is principal with generator $g_0 - 1$.

4. Let R be a ring with identity and $f \in R[[x]]$ be a formal power series with coefficients from R.

   (a) Give a sufficient and necessary condition for f to be a unit in the ring $R[[x]]$. Prove your statement.

   (b) Classify all ideals of $\mathbb{F}[[x]]$ if $\mathbb{F}$ is a field.

METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, SEPTEMBER 2015

SEPTEMBER 28, 2015

1.a) Show that the alternating group A_n has no proper subgroup of index less than n, provided that $n \geq 5$. (Hint: Assume that such a subgroup H exits and then consider the action of A_n on the left cosets of H in A_n.)

b) Use Part (a) to prove that S_n has no proper subgroup of index less than n other than A_n, provided that $n \geq 5$.

2.a) Let p, q be primes with $p \geq q^2$ and G be group of order pq^2. Prove that G has a normal Sylow p-subgroup.

b) In addition to the assumptions in Part (a) assume further that the greatest common divisor $(q^2, p-1) = 1$. Show that the group in Part (a) is abelian.

3.a) Show that the map

$$f : \mathbb{C} \longrightarrow M_2(\mathbb{R}), \qquad f(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

is an injective ring homomorphism, where $M_2(\mathbb{R})$ is the ring of $2x2$-real matrices.

b) Is the image an ideal in $M_2(\mathbb{R})$? Why?

4) Let R be the ring of continuous functions on the interval $[0,1]$.

a) What are the units of the ring R?

b) For any $a \in [0,1]$, let I_a be the set of elements $f \in R$ with $f(a) = 0$. Show that I_a is a maximal ideal in R.

c) Show that the set of elements $f \in R$ with $f(1/2) = 0 = f(1/4)$ is an ideal. Is it prime?

d) Show that any maximal ideal of R is of the form I_a, for some $a \in [0,1]$.

METU Mathematics Department
Graduate Preliminary Examination
Algebra I, February 2016

1. Prove that there is no non-abelian simple group of order 36.

2. If p and q are primes, then show that any group of order p^2q is solvable.

3. Let F_1 be a free group on a nonempty set X_1 and F_2 be a free group on a nonempty set X_2. If $|X_1| = |X_2|$, then show that $F_1 \simeq F_2$.

4. Let R be a ring without identity and with no zero-divisors. Let S be the ring whose additive group is $R \times \mathbb{Z}$ with multiplication defined by

$$(r_1, k_1)(r_2, k_2) = (r_1r_2 + k_2r_1 + k_1r_2, k_1k_2)$$

for any integers $k_1, k_2 \in \mathbb{Z}$ and $r_1, r_2 \in R$. Let

$$A = \{(r, n) \mid rx + nx = 0 \text{ for all } x \in R\}.$$

   (a) Show that A is an ideal in S.
   (b) Show that S/A has an identity and contains a subring isomorphic to R.
   (c) If R is commutative, then show that S/A has no zero-divisors.

5. Let R be a commutative ring with identity such that not every ideal is principal.

   (a) Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.
   (b) If I is an ideal as in (a), show that R/I is a principal ideal ring.

METU Department of Mathematics
PRELIMINARY EXAM
Algebra - I
Feb 13, 2017

Name: ID Number: Signature:

Duration is 3 hours.
Please write your solutions for each question on a separate page.

1.(10+10+5=25 pts.) Let G be a finite group of order $p^n q^2$ where $p > q$ are odd primes and $n \geq 1$.
a) Show that G has a normal subgroup N of index q^2 ($[G : N] = q^2$).
b) Show that G has a normal subgroup K of index q.
c) What are the factors of a composition series of G?

2.(9+8+8=25 pts.) For a group G, the commutator $[x, y]$ of two elements x and y of G is defined as $[x, y] = xyx^{-1}y^{-1}$, and the commutator subgroup G' of G is the subgroup generated by all commutators in G:

$$G' = \langle \{ [x, y] \in G \,|\, x \in G, y \in G \} \rangle.$$

The derived series of G is the sequence of subgroups:

$$G > G^{(1)} > G^{(2)} > ... > G^{(i)} > ...$$

where $G^{(1)} = G'$ and $G^{(i+1)} = (G^{(i)})'$ for $i \geq 1$.
a) Show that for any automorphism f of G we have $f(G^{(i)}) = G^{(i)}$ for any $i \geq 1$.
b) Use part (a) to show that $G^{(i)}$ is a normal subgroup of G for all $i \geq 1$.
c) Show that S_n is not solvable for $n \geq 5$ by explicitly writing down the derived series of S_n.

3.(5+10+10=25 pts.) Let R be a commutative ring with identity 1.
a) Let $M_2(R)$ be the ring of all 2×2 matrices with entries in R. Show that for any ideal I of R, $M_2(I) = \{A \in M_2(R) \,|\, A_{ij} \in I \text{ for all } i, j\}$ is an ideal of $M_2(R)$.
b) Show that any ideal J of $M_2(R)$ is of the form $J = M_2(I)$ for some ideal I of R.
c) Show that if I is an ideal of R, then any prime ideal of the quotient ring R/I is of the form P/I for some prime ideal P of R.

4.(9+8+8=25 pts.) a) Show that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain which is not a principal ideal domain.
b) Let R be a commutative ring with identity 1. Show that if $u \in R$ is a unit and $r \in R$ is a nilpotent element of R (which means $r^n = 0$ for some $n \in \mathbb{Z}^+$), then $u + r$ is a unit in R.
c) Use part (b) to show that if a_0 is a unit in R and a_i is a nilpotent element of R for each $1 \leq i \leq d$, then $f = \sum_{i=0}^{d} a_i x^i$ is a unit in the polynomial ring $R[x]$.