Graduate Preliminary Examination

Algebra II

18.2.2004: 3 hours

Problem 1. Let q be a prime power. Suppose f is an irreducible polynomial of degree m over $\mathbb{F}_q$, and let α be a root of f.

(a) Prove that $\alpha \in \mathbb{F}_{q^m}$.

(b) Prove that α^{q^n} is a root of f in F_{q^m} for all integers n.

(c) Prove that $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{m-1}}$ are distinct roots of f.

Problem 2. Suppose K is an algebraic extension of a field F. Prove that the following are equivalent:

- K is algebraically closed.
- For every algebraic extension L of F, there is an F-monomorphism from L to K.

Problem 3. Let M be a module over a ring R. An element x of M is called **torsion** if $rx = 0$ for some non-zero r in R. Let $T(M)$ be the set of torsion elements of M.

(a) Prove that, if R is an integral domain, then $T(M)$ is a submodule of M, and $M/T(M)$ has no torsion elements.

(b) Find an example where $T(M)$ is not a submodule of M.

Problem 4. Let R be a commutative ring with identity, and let M be a non-zero (unitary) R-module. If $m \in M$, let

$$\operatorname{ord} m = \{r \in R : rm = 0\},$$

and define

$$\mathcal{F} = \{\operatorname{ord} m : m \in M \setminus \{0\}\}.$$

Then $\mathcal{F}$ is partially ordered by $\subseteq$.

(a) Prove that $\operatorname{ord} m$ is an ideal of R.

(b) Prove that every maximal element of $\mathcal{F}$ is a prime ideal.

Graduate Preliminary Examination

Algebra II

18.2.2005: 3 hours

Problem 1. Prove or give a counter-example to the following statement: If M/L and L/K are algebraic extensions of fields, then M/K is algebraic.

Problem 2. Let p be a prime and let $GF(p^m)$ denote the finite field of order p^m.

(a) Show that for any positive integer m, there exists a finite field of order p^m

(b) Show that if $GF(p^m)$ is isomorphic to a subfield of $GF(p^n)$, then m divides n.

(c) Let E be the algebraic closure of $GF(p)$. Show that there is an intermediate field L between $GF(p)$ and E with $|L : GF(p)| = \infty$ and $|E : L| = \infty$

Problem 3. Let R be a commutative ring with identity , I an ideal of R and $L = \{a \in R : \ aI = 0\}$

(a) Prove that each $a \in L$ induces an R-module homomorphism

$$\bar{\lambda}_a : R/I \to R$$

(b) Using (a), prove that the R-modules L and $Hom_R(R/I, R)$ are isomorphic.

Problem 4. Let R be a commutative ring with identity, and let M be a unitary R-module. Then M is called:

- **torsion-free**, if $r \cdot m = 0$ implies either $r = 0$ or $m = 0$ where r in R and m in M;
- **divisible**, if for all m in M and non-zero r in R, there is n in M such that $r \cdot n = m$.

Assume M is torsion-free and non-trivial.

(a) Prove that R is an integral domain. Show that, the hypothesis that M is non-trivial is necessary.

Now let K be the quotient-field of R.

(b) Prove that $K \otimes_R M$ is torsion-free as a K-module.

(c) Prove that, if M is divisible, then $\phi : m \mapsto 1 \otimes m$ is an R-module epimorphism from M onto $K \otimes_R M$.

METU - Department of Mathematics
Graduate Preliminary Exam

Algebra II

February, 2009

Duration: 180 min.

1. Let α be the real positive 16th root of 3 and consider the chain of intermediate fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\alpha^8) \subseteq \mathbb{Q}(\alpha^4) \subseteq (\alpha^2) \subseteq \mathbb{Q}(\alpha) = F.$$

a) Compute the degrees of these five intermediate fields over $\mathbb{Q}$ and conclude that these fields are all distinct.

b) Show that every intermediate field between $\mathbb{Q}$ and F is one of the above. (Hint: If $\mathbb{Q} \subseteq K \subseteq F$, consider the constant term of the minimal polynomial of α over K).

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2. Let p be a prime number and let $w_p = e^{2\pi i/p}$ be the pth root of 1 in $\mathbb{C}$.

a) Show that Gal $(\mathbb{Q}(w_p)/\mathbb{Q})$ is isomorphic to the multiplicative group $\mathbb{Z}_p^*$.

b) Let F be a field containing w_p and let a be an element of F which is not the pth power of any element of F. Show that if E is the splitting field of the polynomial $x^p - a \in F[x]$, then Gal (E/F) is isomorphic to the additive group $\mathbb{Z}_p$.

c) If K is the splitting field of $x^p - 2 \in \mathbb{Q}[x]$, show that $|K : \mathbb{Q}| = p(p-1)$.

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3. Let R be a ring. Recall that an R-module P is called projective if for every R-module epimorphism $f : A \to B$ and every R-module homomorphism $g : P \to B$, there exists an R-module homomorphism $h : P \to A$ such that $fh = g$.

a) Let P be an R-module for a ring R. Show that if there is a free R-module F and an R-module K such that $F \cong K \oplus P$, then P is projective. (You may use the fact that every free module is projective).

b) Let R be a commutative ring. Suppose that R-modules P and Q are projective. Show that $P \otimes_R Q$ is projective.

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4. Let R be a ring with unity and suppose that R can be written as the sum $R = \sum_{i=1}^{m} I_i$, where I_i are finitely many (two-sided) ideals of R satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.

a) Prove that, for every simple right R-module M, there exists a unique subscript k such that $MI_k \neq 0$

b) Show that if $i \neq j$, then every right R-module homomorphism $\theta : I_i \to I_j$ is the zero map.

GRADUATE PRELIMINARY EXAMINATION

ALGEBRA II
Spring 2010

1. Let R be a commutative ring with identity 1 and let Q be an injective R-module. If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence of R-modules and R-homomorphisms with the property that $f \circ \alpha = 0$ for an R-homomorphism $f : M \longrightarrow Q$, show that there is an R-homomorphism $g : N \longrightarrow Q$ with $g \circ \beta = f$.

2. A nonzero left module M (over some ring) is called
• **simple**, if M has no proper nonzero submodule;
• **complemented**, if every submodule of M is a direct summand of M (that is, for every submodule A of M, there is a submodule B of M such that $M = A \oplus B$, which means $M = A + B$ and $A \cap B = 0$).

(a) Give an example of a simple module.

(b) Give an example of a complemented module that is not simple.

(c) Show that every nonzero submodule of a complemented module is complemented.

(d) Show that every complemented module has a simple submodule.

3. Suppose K, L, and M are fields, and $K \subseteq L \subseteq M$. Prove or disprove the following statements.

(a) If M/L and L/K are normal, then so is M/K.

(b) If M/K is normal, then so is M/L.

(c) If M/L is normal, then so is M/K.

(d) $(K, +) \ncong (K^*, \cdot)$.

4. Consider the polynomial $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$

(a) Using Eisenstein's criterion, prove that f is irreducible over $\mathbb{Q}$.

(b) Let E be the splitting field of f. Show that there exists $\sigma \in Gal(E/Q)$ of order 5.

(c) Prove the following:

There exists $\tau \in Gal(E/Q)$ of order 2 and hence $Gal(E/Q) \cong S_5$.

(Hint : You may assume that $f(x)$ has exactly one pair of complex conjugate roots.)

(d) Is $f(x)$ solvable by radicals over $\mathbb{Q}$? Why?

TMS EXAM
February 16, 2012
ALGEBRA II

1. Prove that there is no field F such that $F^+ \cong F^*$.

2. Let F be a field with 16 elements.

   (i) Show that there exists an element $\alpha \in F$ with $\alpha^4 = \alpha + 1$.

   (ii) Find the factorization of $x^3 + x + 1 \in F[x]$ into irreducible polynomials over F.

   (iii) Find all subfields of F.

   (iv) Does there exist a quadratic, irreducible polynomial over F ? Explain your reasoning.

3. Let R be a commutative ring with unity, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   be an exact sequence of R-modules. Let $r, s \in R$ be such that $(r, s) = R$. Suppose that $rA = sC = 0$.

   (i) Show that the map $\alpha : C \longrightarrow C$ given by $\alpha(c) = rc$ is an isomorphism.

   (ii) Show that $g|_{rB}$ is an isomorphism between rB and C.

   (iii) Show that $B \cong A \oplus C$.

4. Let R and S be two rings, A a right R-module, C a right S-module and B an (R, S)-bimodule. Show that $\mathrm{Hom}_S(A \otimes_R B, C)$ and $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$ are isomorphic abelian groups.

GRADUATE PRELIMINARY EXAMINATION
ALGEBRA II, FEBRUARY 2013

FEBRUARY 14, 2013

1.a. Show that $f(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$ is the unique irreducible polynomial of degree three so that sum of its roots in a splitting field is equal to zero, where $\mathbb{F}_2$ denotes the field of two elements.

1.b. Show that $E = \mathbb{F}_2[x]/(f(x))$ is the splitting field of $f(x)$, by finding all zeros of $f(x) = x^3 + x + 1$ in E.

1.c. Show that the extension $\mathbb{F}_2 \subseteq E$ is Galois, by determining its Galois group. Describe the action of the elements of the Galois group.

2.a. Construct a Galois field extension K of the field of rational numbers, so that $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ and the Galois group is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.b. Determine the intermediate fields of this extension.

2.c. Show that $\mathbb{Q}(\sqrt[4]{2})$ is a degree four extension of $\mathbb{Q}$, which is not Galois.

3.a. Let M denote the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Determine the quotient module M/N, where N is generated by the vectors $(1, 2, 3)$, $(0, -2, 5)$, $(2, 0, 8)$ and $(0, 1, 2)$. (Use Smith Normal From.) Calculate the number elements of the quotient module if it is finite.

3.b. Let $\mathbb{F}$ be a field, $R = \mathbb{F}[x]$ the polynomial ring over $\mathbb{F}$, and $M = R \oplus \cdots \oplus R$ the free R-module of rank n, for some positive fixed integer n. For any $n \times n$-matrix A with entries in the field $\mathbb{F}$, consider the assocaited matrix $A - xI_{n\times n}$, with entries in the ring R, where $I_{n\times n}$ is the $n \times n$-identity matrix. Let N_A be the submodule of M generated by the rows of $A - xI_{n\times n}$. Calculate the quotient modules, M/N_{A_i}, using Smith Normal Form again, for the matrices

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

with entries in the field of rational numbers (i.e., we take $\mathbb{F} = \mathbb{Q}$ and $n = 3$). Are the quotient modules isomorphic? Relate the quotient modules to the characteristic and the minimal polynomial of the matrices.

4. Let R be an integral domain and M an R-module so that for any $r \neq 0 \in R$ and $m \neq 0 \in M$, we have $rm \neq 0$. The R-module M is called divisible if for each $m \in M$ and nonzero element $r \in R$ there exists an element $m' \in M$ such that $m = rm'$.

4.a. Prove that the direct sum of two divisible R-modules is also divisible.

4.b. An R-module M is called injective, if whenever $i : M \to N$ is an embedding of R-modules and $\phi : M \to L$ is an R-module homomorphism then there is an R-module homomorphism $\Phi : N \to L$ so that $\phi(m) = (\Phi \circ i)(m)$, for all $m \in M$. Prove that an injective R-module is divisible.

METU Mathematics Department
Graduate Preliminary Examination
Algebra II, January 2014

1. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Note that the ring $\mathbb{Z}/2\mathbb{Z} \cong R/I$ is naturally an R-module.

   - Show that the map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by

     $$\phi(a_0 + a_1 x + \ldots + a_n x^n, b_0 + b_1 x + \ldots + b_m x^m) = \frac{a_0}{2} b_1 \pmod 2$$

     is R-bilinear.
   - Show that $2 \otimes x - x \otimes 2$ is nonzero in $I \otimes_R I$.

2. Let I_n be the identity matrix of dimension n.

   - Prove that there is no 3×3 matrix A over $\mathbb{Q}$ such that $A^8 = I_3$ but $A^4 \neq I_3$.
   - Write down a 4×4 matrix B over $\mathbb{Q}$ such that $B^8 = I_4$ and $B^4 \neq I_4$.

3. Let $\mathbb{F}_q$ be a finite field with q elements and let $K/\mathbb{F}_q$ be a quadratic extension.

   - For any $\alpha \in K$, show that $\alpha^{q+1} \in \mathbb{F}_q$.
   - Show that every element of $\mathbb{F}_q$ is of the form β^{q+1} for some $\beta \in K$.

4. Let p be an odd prime and let $L = \mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field.

   - Show that L has a unique subfield K such that $[K : \mathbb{Q}] = 2$.
   - If $p = 5$, then find an element $\alpha \in L$ such that $L = K(\alpha)$ and $\alpha^2 \in K$.
   - If $p \geq 7$, then show that there is no $\alpha \in L$ such that $L = K(\alpha)$ and $\alpha^{(p-1)/2} \in K$.

Graduate Preliminary Examination

Algebra II

22.9.2004: 3 hours

Problem 1. Let $K_0 = \mathbb{F}_{11}[X]/(X^2+1)$ and $K_1 = \mathbb{F}_{11}[Y]/(Y^2+2Y+2)$.

(a) Show that the K_i are fields for $i = 0, 1$.

(b) Find the orders of the K_i for $i = 0, 1$.

(c) Either exhibit an isomorphism of K_0 and K_1, or show that they are not isomorphic.

Problem 2. Show that the sum of all elements of a finite field is zero, except for $\mathbb{F}_2$.

Problem 3. Let L/K be a field-extension, and let α be algebraic over K with minimal polynomial f. Let $M = K(\alpha) \otimes_K L$. We know that M is a vector-space over L.

(a) Exhibit an embedding of L in M (as vector-spaces over L).

(b) Exhibit an embedding ι of L in M and a multiplication $\cdot$ on M such that the following conditions hold:

- M is a commutative ring with identity;
- ι is a ring-homomorphism;
- if $m \in M$ and $\ell \in L$, then $\iota(\ell) \cdot m$ is the product ℓm given by the vector-space structure.

(c) Show that $L[X]/(f)$ and $K(\alpha) \otimes_K L$ are isomorphic as rings.

Problem 4. Let R be a ring with 1. If M is an R-module, the **uniform dimension** of M ($\operatorname{ud} M$) is the largest integer n such that there is a direct sum $M_1 \oplus \ldots \oplus M_n \subseteq M$ with all the M_i non-zero. If no such integer exists then we say that $\operatorname{ud} M = \infty$. If $M \subseteq N$ are R-modules, M is said to be **essential** in N if every non-zero submodule of N has non-zero intersection with M. Suppose the $\operatorname{ud} M < \infty$ and $M \subseteq N$. Prove that M is essential in N if and only if $\operatorname{ud} M = \operatorname{ud} N$

PRELIMINARY EXAMINATION
ALGEBRA II
FalI 2005
September 16^{th}, 2005

Duration: 3 hours

1. Let $f(x) = x^3 - 2x - 2 \in \mathbb{Q}[x]$. Let $K = \mathbb{Q}(\alpha)$ where α is a real root of f, and let F be the Galois closure of the extention $K/\mathbb{Q}$.

a) Determine the group of $\mathbb{Q}$-automorphisms of K.

b) Determine the Galois group $G(F/\mathbb{Q})$.

c) Determine the Galois group $G(F/K)$.

2. Let K be a field of characteristic p (where p is a prime number). Let $K^p = \{b^p | b \in K\}$.

a) Show that K^p is a subfield of K and K/K^p is an algebraic extension.

b) Let $a \in K, a \notin K^p$. Prove that $[K^p(a) : K^p] = p$.

3. Let R be a principal ideal domain, M a free R-module, and S a submodule of M. S is called a pure submodule if

whenever $ay \in S$ (with $a \in R \setminus \{0\}, y \in M$), then $y \in S$.

a) Show that $\{0\}$ and R are the only pure submodules of R, considered as an R-module)

b) Find a proper, nontrivial pure submodule of $R \oplus R$ (considered as an R-module).

c) Let N be a torsion-free R-module and $\varphi : M \to N$ be an R-module homomorphism. Prove that $Ker\varphi$ is a pure submodule of M.

4. Let R be a commutative ring with identity. Prove that every submodule of R is free iff $R = \{0\}$ or R is a principal ideal domain. (**Warning:** To prove that R is a PID, you have to show R is an integral domain first.)

TMS

Fall 2009

ABGEBRA II

1. Let R be a commutative ring with identity, and $A, B, C_1, \cdots, C_n$ be R-modules.

   a) Assume A is a submodule of B, and B satisfies the ACC on its submodules. Show that B/A satisfies the ACC on its submodules.

   b) Assume $C_1, \cdots, C_n$ satisfy the ACC on their submodules. Show that their direct sum also satisfies the ACC on its submodules.

   c) If the ring R satisfies the ACC on its ideals, then show that every finitely generated R-module satisfies the ACC on its submodules.

2. Let M be a left R-module.

   a) Prove that M is a simple module if and only if $M = Rm$ for all nonzero $m \in M$.

   b) Prove that M is simple if and only if $M \cong R/I$ for a maximal left ideal $I \subseteq R$.

   c) Prove that if M is simple, then $\mathrm{End}_R(M)$ is a division ring.

3. Let K be a subfield of a finite field L. Describe (as precisely as possible) the group of automorphisms of L when it is considered as:

   a) a field

   b) a vector space over K,

   c) an additive group

4. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

   a) Find the order of the Galois group G_f of $f(x)$ over $\mathbb{Q}$.

   b) Show that G_f is isomorphic to the group H given by generators a of order 5 and b of order 4, with the relation $ba = a^2b$.

TMS

Fall 2010

Algebra II

1. Let R be a principal ideal domain, and let M be a finitely generated module over R. We know that, for some non-negative integers n and s, there are nonzero non-units $q_1, \ldots, q_n$ of R such that $q_k \mid q_{k+1}$ and M$\cong R/(q_1) \oplus \cdots \oplus R/(q_n) \oplus R^s$.

   (a) Letting K be the quotient field of R, find the dimension of $M \otimes_R K$ as a vector space over K.

   (b) Find the greatest integer t such that M has linearly independent elements $x_1, \ldots, x_t$: this means, if $a_1, \ldots, a_t \in R$ and $a_1x_1 + \cdots + a_tx_t = 0$, then $a_1 = \cdots = a_t = 0$.

   Suppose further that R has only one prime ideal different from $\{0\}$, namely (p).

   (a) Give an example of such a ring R.

   (b) Show that $R/(p)$ is a field.

   (c) Show that, for some integers k_i such that $0 < k_1 < \cdots < k_m$, M$\cong R/(p^{k_1}) \oplus \cdots \oplus R/(p^{k_n}) \oplus R^s$.

   (d) Letting L be the field $R/(p)$, find the dimension of M/pM as a vector space over L.

   (e) Find the least integer t for which some subset $\{x_1, \ldots, x_t\}$ of M generates M over R.

   (f) Letting L be the field $R/(p)$, find the dimension of M/pM as a vector space over L.

   (g) Find the least integer t for which some subset $\{x_1, \ldots, x_t\}$ of M generates M over R.

2. Suppose E and F are finite extensions of a field K, and E and F are themselves subfields of some large field, so that the compositum EF is well defined: EF
E@-[ur]F@-[ul]
K@-[ul]@-[ur] Let us say that E is **free** from F over K if any elements of E that are linearly independent over K are still linearly independent (as elements of EF) over F.

   i. If E is free from F over K, show that F is free from E over K.

   ii. Prove that the following are equivalent:

      A. E is free from F over K,

      B. $[E : K] = [EF : F]$,

C. $[E : K][F : K] = [EF : K]$.

Suppose now also that E/K is Galois.

i. Prove that EF/F is Galois.

ii. Prove that E is free from F over K if and only if $E \cap F = K$.

3. Let p be a prime, $q = p^t$ for some $t \geq 1$, $F(q^k)$ denote the field with q^k elements, and $L(q) = \cup_{n \geq 1} F(q^{n!})$.

   (a) Show that $L(q)$ is a field. What is its prime subfield?

   (b) Show that $L(q)$ is an algebraic extension of $F(q)$.

   (c) Is $L(q)$ algebraically closed?

4. Let U be a right R-module and $X \subseteq U$ be any subset. Then show that

   (1) $ann_R(X) = \{r \in R | xr = 0 \quad \forall x \in X\}$ is a right ideal of R

   (2) If X is an R-submodule of U, then $ann_R(X)$ is an ideal of R.

   (3) If U is simple and $0 \neq x \in U$, then $ann_R(x)$ is a maximal right ideal of R and $U \cong R^{\bullet}/ann_R(x)$ where $R^{\bullet}$ denotes R as an R-module.

M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

ALGEBRA II

Duration : 3 hr.

Each question is 25 pt.

1. Let n be a positive integer and F be a field of characteristic p with $p \not| n$.

   Let $f(x) = x^n - a$ for some $0 \neq a \in F$ and E be a splitting field for $f(x)$ over F.

   a) Show that $f(x)$ has no multiple roots (that is $f(x)$ has n distinct roots.)

   b) Show that E contains a primitive n-th root of unity ϵ.

   c) Assume that $\epsilon \in F$. Show that all irreducible factors of factors of $f(x)$ in $F[x]$ have the same degree and $[E : F]$ divides n.

2. Let α be an element of $\mathbb{C} - \overline{\mathbb{Q}}$ where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

   a) Show that $\mathbb{Q}(\alpha)$ is the field of fractions of the integral domain $\mathbb{Q}[\alpha]$.

   (Hint : Use the homomorphism $\mathbb{Q}[x] \to \mathbb{C}$, $x \mapsto \alpha$).

   b) Show that

   - each matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Q})$ defines an automorphism

   $$\Phi_M : \mathbb{Q}(\alpha) \longrightarrow \mathbb{Q}(\alpha) \text{ given by } \alpha \mapsto \frac{a\alpha + b}{c\alpha + d}$$

   - and we obtain a group homomorphism $\Psi : GL(2, \mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathbb{Q}(\alpha))$, $\Psi(M) = \Phi_M$.

   c) True or false? Explain.
   Ψ in (b) is an isomorphism.

3. Let M be a module over a commutative ring R satisfying the descending chain condition.

   Suppose that f is an endomorphism of M. Show that f is an isomorphism if and only if f is a monomorphism.

4. Let R be commutative ring with unity and M be an R-module. For $x \in M$ we define

$$\text{Ann}(x) = \{r \in R : rx = 0\}$$

   and we set $T(M) = \{x \in M : \text{Ann}(x) \neq 0\}$.

   a) Show that

   - If $x \neq 0$, then $\text{Ann}(x)$ is a **proper ideal** in R.
   - If for each maximal ideal $\mathbf{p}$ in R there exists some $r \in \text{Ann}(x)$, $r \notin \mathbf{p}$, then $x = 0$.

   b) Let R be an integral domain with field of fractions F. Show that

   - $T(M)$ is a submodule of M.
   - $T(M)$ is in the kernel of the map $M \to M \otimes_R F$, $\quad m \mapsto m \otimes 1$.
   - $T(M) = \{0\}$ if M is a **flat** R-module.

   c) True or false ? Prove the statement or give a counter example.

   For any R and an R-module M, $T(M)$ is a submodule of M.

METU MATHEMATICS DEPARTMENT
ALGEBRA II
SEPTEMBER 2012 - TMS EXAM

1. Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, and let α be a root of $f(x)$ in $\mathbb{C}$.

a) Find the splitting field E of $f(x)$ over $\mathbb{Q}$.

b) Find the degree of the extension E over $\mathbb{Q}$.

c) Find the automorphism group G of E over $\mathbb{Q}$. Find the lattice of subgroups of G.

d) Choose one of nontrivial proper subgroups of G and find the intermediate field corresponding to this subgroup explicitly.

2. a) Prove that for a finite field F of characteristic p, the map $u \longmapsto u^p$ is an automorphism of F.

b) For every integer n, show that the map $u \longmapsto u^4 + u$ is an endomorphism of the additive group of the finite field $\mathbb{F}_{2^n}$, and determine the size of the kernel and image of this endomorphism.

3. Let R be a ring with 1, and let N be a submodule of an R-module M.

a) Prove that M is torsion if and only if N and M/N are both torsion.

b) Prove that if N and M/N are both torsion-free, then M is torsion-free. Give an example to show that the converse of this statement is false.

c) Prove that a free module over a PID is torsion-free. Give an eample to show that the converse of this statement is false.

4. Let R be a ring with 1 and let M, N be R-modules.

a) Prove that if M and N are both free, then $M \otimes_R N$ is free.

b) Let $f : M \to N$ be an R-homomorphism and let W be an R-module. Show that if f is surjective, then the induced map $f \otimes 1_W : M \otimes_R W \to N \otimes_R W$ is surjective. Give an example of M, N, W and an injective map $f : M \to N$ to show that the induced map $f \otimes 1_W$ is not injective.

METU-MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA II, SEPTEMBER 2013

SEPTEMBER 17, 2013

1) Suppose that $f(x)$ is irreducible in $F[x]$ and K is a Galois extension of F. Show that all irreducible factors of $f(x)$ in $K[x]$ have the same degree.

2.a) Find the minimal polynomial of $i\sqrt{5}+\sqrt{2} \in \mathbb{C}$ over the rational numbers. Determine the Galois group of the splitting field of the minimal polynomial over the field of rational numbers.

2.b) Find a primitive element over the field of rational numbers for the extension field $K = \mathbb{Q}(\sqrt{5}, \sqrt[3]{4})$.

3.a) Suppose R is a commutative ring and M is an R-module. A submodule N is called pure if $rN = rM \cap N$, for all $r \in R$. Show that any direct summand of M is pure.

3.b) If M is torsion free and N is a pure submodule show that M/N is torsion free.

3.c) If M/N is torsion free show that N is pure.

4.a) Prove that any finitely generated projective module M over a PID R is free.

4.b) Is $\mathbb{Z}$-module of rational numbers $\mathbb{Q}$ projective? What about the $\mathbb{Q}$-module of rational numbers $\mathbb{Q}$?

METU Mathematics Department
Graduate Preliminary Examination
Algebra II, Fall 2014

1. Let A be an abelian group considered as a $\mathbb{Z}$-module. If A is finitely generated than show that $A \otimes_{\mathbb{Z}} A \cong A$ if and only if A is cyclic. Is the same statement true if A is not finitely generated?

2. Let $T : V \to W$ be a linear transformation of vector spaces over a field $\mathbb{F}$.

   (a) Show that T is injective if and only if $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set in W for every linearly independent set $\{v_1, \ldots, v_n\}$ in V.

   (b) Show that T is surjective if and only if $\{T(x) : x \in X\}$ is a spanning set for W for some spanning set X for V.

   (c) Let $D : \mathbb{F}[x] \to \mathbb{F}[x]$ be the derivative map on polynomials, i.e. $D(f(x)) = f'(x)$, which is a linear transformation. Investigate if D is injective, surjective using the previous parts.

3. Let K be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$.

   (a) Show that $\sqrt{-1}$ is an element of K.

   (b) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_8.

   (c) Compute the lattice of subfields of K.

4. Let $\mathbb{F}_q$ be a finite field of order $q = p^n$ for some prime number p. Show that the set of subfields of $\mathbb{F}_q$ is linearly ordered (i.e. $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$ for every pair of subfields.) if and only if n is a prime power.

METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA II, SEPTEMBER 2015

SEPTEMBER 29, 2015

1.a) Let R be a commutative ring with unity. A module P over R is called projective, if for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $f \circ h = g$. Prove that every free R-module P is projective.

b) Show more generally that an R-module P is projective if and only if there is an R-module N such that $P \oplus N$ is a free R-module.

c) Show that a finitely generated projective $\mathbb{Z}$-module P is indeed free. (This part of the question can be answered independently from the previous parts.)

2) Let A be a commutative ring with unity and M is a finitely generated A-module. Assume that $f : M \to A^n$ is a surjective homomorphism. Show that $\ker(f)$ is also finitely generated. (Hint: Choose a basis $\{e_1, \cdots, e_n\}$ for A^n, and let $m_i \in M$ with $f(m_i) = e_i$. Show that M is isomorphic to the direct sum of $\ker(f)$ and the submodule generated by $m_1, \cdots, m_n$)

Is it true that a submodule of a finitely generated module is finitely generated?

3.a) Find the splitting field K of the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$.

b) Determine the Galois group of the extension $K/\mathbb{Q}$.

c) Show that $\sqrt[3]{2}$ cannot be written as a $\mathbb{Q}$-linear combination of n^{th} roots of unity for any positive integer n.

4.a) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial and let $G = Gal(K : \mathbb{Q})$ be the Galois group of its splitting field K. Considering G as a subgroup of S_5, the symmetric group on five letters, show that G contains a five cycle.

b) Assume further that exactly three roots of $f(x)$ are real. Show that the Galois group G contains a transposition.

c) Is the polynomial $f(x)$ solvable by radicals? (Hint: It is a fact that any subgroup of S_5 that contains a 5-cycle and a transposition is equal to S_5).

METU Mathematics Department
Graduate Preliminary Examination
Algebra II, February 2016

1. Let R be an integral domain.

   (a) Show that free R-modules are torsion free.

   (b) Exhibit an R-module which is finitely generated and torsion free but not free.

   (c) Suppose that R is a principal ideal domain and let M be a finitely generated R-module. Show that M is torsion free if and only if M is free.

2. If $f : A \to A$ is an R-module homomorphism such that $f \circ f = f$, then show that

$$A = \operatorname{Ker}(f) \oplus \operatorname{Im}(f).$$

3. Let R be a commutative ring with identity. Let I and J be ideals of R. Prove that the R-module $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I+J)$.

4. Let K be the splitting field of the polynomial $f(x) = x^6 + 3$ over $\mathbb{Q}$. Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to S_3.

5. Let F be a field.

   (a) Suppose that K is an algebraic extension of F and R is a ring such that $F \subseteq R \subseteq K$. Show that R is a field.

   (b) Suppose that $L = F(x)$, i.e. the field of rational functions over F. If $u \in L \setminus F$, then show that u is transcendental over F.