16 February 2004

Graduate Preliminary Examination
Complex Analysis
Duration: 3 hours

1. Calculate $\int_\gamma \frac{w^3+1}{w(w^4+w^3+2w^2+1)}dw$, where C is the circle of radius 1/3 centered at 0, traced once in the clockwise direction.

2. Let $\mathbb{D} = \{z : |z| < 1\}$. Prove that there is **no** holomorphic function $f : \mathbb{D} \to \mathbb{D}$ satisfying $f(\frac{i}{2}) = 0$, $f(\frac{1}{3}) = 0$, and $f(0) = \frac{1}{5}$.

3. Suppose f is holomorphic on $\{z : |z| < 1\} = \mathbb{D}$. Prove that there is a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.
(Hint: Consider the zeroes of f.)

4. Show that $\prod_{n=0}^{\infty}(1+z^{(2n)}) = \frac{1}{1-z}$ for each $z \in \mathbb{D} = \{z : |z| < 1\}$. Prove that convergence is uniform on compact subsets of $\mathbb{D}$, but not uniform on $\mathbb{D}$.

GRADUATE PRELIMINARY EXAMINATION
ANALYSIS 2 (Complex Analysis)

February 17th, 2005

1. Let G be the group of analytic automorphisms $g : D(0:1) \to D(0:1)$ of the open unit disc D(0:1) onto itself.

   (a) For any two elements z_1, z_2 in D(0:1), explicitly construct $g \in G$ such that $g(z_1) = z_2$.

   (b) Characterize the elements of $G_0 = \{g \in G : g(0) = 0\}$.

   (c) Determine all holomorphic functions $f : D(0:1) \to \mathbb{C}$ which are G-invariant, i.e. $f(g(z)) = f(z) \forall g \in G \;, z \in D(0:1)$.

   (d) Determine all holomorphic functions $f : D(0:1) \to \mathbb{C}$ which are G_0-invariant.

2. Let f be an entire function which satisfies

$$f(z) + f(z+1) = f(2z) \;\; \forall z \in \mathbb{C}.$$

   (a) Using induction on n show that

$$f(2^n z) = \sum_{k=0}^{2^n-1} f(z + \frac{k}{2^{n-1}}) \; \forall n \in \mathbb{Z}, n \geq 1$$

   (b) Let D(0,r) denote the open unit disc with center at $0 \in \mathbb{C}$ and radius $r > 0$. Using the Cauchy Integral Formula over the counterclockwise oriented circle of radius 2^n centered at 0 or otherwise, show that for any $a \in D(0,1)$ and $n \in \mathbb{Z}, n \geq 1$

$$|f''(a)| \leq \frac{M}{2^{n-4}}$$

   where $M = \sup_{z \in D(0:3)} |f(z)|$.

   (c) Prove that $f(z) = Az + B$ for some $A, B \in \mathbb{C}$ with A+B=0.

3. Consider the series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

   (a) Show that f(z) defines an analytic function in the open unit disc D(0,1).

   (b) Verify that for all $k \geq 1$, and for all k-th roots of unity w, (i.e. $w^k = 1$), $f(w) = \infty$ holds.

   (c) Show that in any arc on the unit circle $|z| = 1$, there are N-th roots of unity for infinitely many N. (*Hint: Use the map* $[0,1] \to \{z : |z| = 1\}, t \longmapsto e^{2\pi i t}$ *to work in* [0,1]).

   (d) Using the results of b) and c) show that f(z) cannot be continued analytically to any domain Ω which properly contains D(0,1).

4. Let Ω be a convex bounded domain and γ a closed smooth curve in Ω. Suppose that f and g are analytic functions on $\overline{\Omega}$, f zero free on γ.

   (a) Compute the residue of $\frac{g.f'}{f}$ at a zero of f in Ω.

   (b) Compute $\frac{1}{2\pi i} \oint_\gamma \frac{g(z)f'(z)}{f(z)} dz$.

METU - Mathematics Department
Graduate Preliminary Exam-Spring 2008

Complex Analysis

NOTATION :

$D = \{z \in \mathbb{C} : |z| < 1\}$.
Unless otherwise stated, Ω denotes an open connected set in $\mathbb{C}$.
For a region Ω, $\text{Aut}(\Omega)$ denotes the group of holomorphic automorphisms of Ω.

1. A) Let $f : \Omega \to \mathbb{C}$ be a function with

$$f(z) = u(x,y) + iv(x,y)$$

for any $z = x + iy \in \Omega$. Prove that for each $\alpha + i\beta \in \Omega$ at which $f(z)$ is differentiable as a function of z, the functions u, v have partial derivatives at (α, β) which satisfy the Cauchy-Riemann equations.

B) Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = x^3 + i(1-y)^3$$

is differentiable only at $z = i$. Evaluate $f'(i)$.

C) Prove that the real and imaginary parts of $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

satisfy the Cauchy-Riemann equations at $(0,0) \in \mathbb{R}^2 \cong \mathbb{C}$ but f is not differentiable at $0 \in \mathbb{C}$.

2. A) Let $f : \Omega \to \mathbb{C}$ be an analytic function. Given $a \in \Omega$, prove that for $m \in \mathbb{N}$

$$\text{Res}_{z=a}\left(\frac{f(z)}{(z-a)^{m+1}} \right) = \frac{1}{m!} \frac{d^m f(z)}{dz^m} \Big|_{z=a}.$$

B) Compute

$$\mathrm{Res}_{z=i}\left(\frac{e^{iz}}{(z^2+1)^2}\right).$$

C) Prove that

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)^2}dx=\frac{\pi}{e}\quad .$$

3. Let $z_1, z_2 \in D$ be any two distinct points and let $\Omega = D - \{z_1, z_2\}$.

   a) Show that any analytic map $f : \Omega \to D$ extends to an analytic map $D \to D$.

   b) Show that in part (a) if f is an isomorphism, then so is the extension.

   c) Find a relation between z_1 and z_2 which is necessary and sufficient in order to have an isomorphism $\Omega \to D - \{0, 1/2\}$.

4. True or false ? Prove the statement or give a counter example.

   a) If $f(z)$ is a non-constant entire function such that $|f(z)|$ is bounded on $\mathbb{R}$, then $f(z)$ has an essential singularity at ∞.

   b) If $f(z)$ is meromorphic in $\mathbb{C} \cup \{\infty\}$, then it is a rational function (ie. the ratio of two polynomials).

   c) Let $p(z)$ be a polynomial such that for all sufficiently large R we have

$$\int_{|z|=R}\frac{p'(z)}{p(z)}dz = 2\pi i N, \text{ for some } N \geq 1.$$

Then $p(z)$ defines a surjective holomorphic mapping $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of degree N.

M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

COMPLEX ANALYSIS

Duration : 180 min.

1. (25 pt.) Consider the entire function $f(z) = e^{z^2}$.

   a) Show that for $w \in \mathbb{C}^*$ the set $f^{-1}(w) = \{z : f(z) = w\} \neq \emptyset$ and is discrete.

   b) Can you find $w \in \mathbb{C}^*$ for which the set $f^{-1}(w) = \{z : f(z) = w\}$ is bounded ? How is your answer related to the behaviour of $f(z)$ at ∞ ?

   c) Show that there exists a disc $D(0;\delta)$ such that $f(z)$ takes each value $w \in f(D(0;\delta))$ exactly twice in $D(0;\delta)$.

2. (25 pt.) Consider the set $\mathcal{F}$ of all meromorphic functions on $\mathbb{C}$ which have exactly the following zeros and poles.

   Zeros at $z_1 = 0, z_2 = 1$ of order 2, 3 respectively, poles at $p_1 = i, p_2 = -i$ each of order 2.

   a) Write a rational function $f_0(z) \in \mathcal{F}$.

   b) Determine the structure of the most general function $g(z) \in \mathcal{F}$.

   c) For an arbitrary function $g(z) \in \mathcal{F},\ g \neq f_0$ determine the type of the singularity at each singular point in $\mathbb{C}$ of $f_0(z).g(z),\ f_0(z) + g(z)$.

3. (25 pt.) Let $D^*(a;R)$ denote the disc of radius $R > 0$ with a puncture at the center a.

   a) Write an analytic isomorphism $\Phi : D^*(0,R_1) \to D^*(i;R_2)$.

   b) Prove that every analytic isomorphism $\Phi : D^*(0,R_1) \to D^*(i;R_2)$ extends to an analytic isomorphism $D(0,R_1) \to D(i;R_2)$.

   c) Using Φ you wrote in (a), construct an analytic isomorphism

$$\Psi : D(0;R_1) - \{1/2\} \to D^*(i;R_2).$$

4. (25 pt.) True or false ? Explain.

a) Suppose that $f(z)$ is analytic in $D(0;\delta)$ and let $g(z) = (f(z)-1)^N$ for some integer $N \geq 1$. If

$$\int_{\Gamma(r)} \frac{dg(z)}{g(z)} = 6\pi i N \text{ for all circles } \Gamma(r) : |z| = r, 0 < r < \delta$$

then $f(0) = 1,\ f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$.

b) If $g(z)$ is analytic in $\Omega = \mathbb{C} - \{a, b\}$ and satisfies

$$\text{Residue}(g; a) = \text{Residue}(g; b) = 0$$

then for $z \in \Omega$ the integral $F(z) = \displaystyle\int_0^z g(u)du$ is independent of the path connecting 0 and z.

c) The function $f(z) = \sin(\sqrt{z^2-1})$ has an analytic branch in

$$\mathbb{C} - \{z \in \mathbb{R} : z \geq 1\}.$$

d) There exist non-constant doubly periodic functions $f(z)$ with simple poles at each point of the period lattice $L = \{m + ni : m, n \in \mathbb{Z}\}$.

PRELIMINARY EXAM - Feb.2012
Complex Analysis

Duration : 3 hr.

Q.1	Q.2	Q.3	Q.4	Total

1. (5 + 7 + 13 = 25 pt.) Let $n \in \mathbb{N}$ with $n \geq 2$ and $\omega = e^{\pi i/n}$.

   (A) Show that $\omega^{\frac{n(n-1)}{2}} = i^{n-1}$.

   (B) Show that $\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1} \left(x - \omega^{2k}\right)$ for every $x \neq 1$.

   (C) Prove that

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} & \text{if} \quad n \text{ is odd} \end{cases} .$$

2. (4 + 8 + 13 = 25 pt.)

   (A) Prove that $|e^z| = e^{\mathrm{Re}(z)}$

   (B) Let f be an entire function such that $|f(z)| \leq e^{\mathrm{Re}(z)}$. Show that there exists a constant $a \in \mathbb{C}$ such that

$$f(z) = ae^z .$$

   (C) Let g be an entire function such that $g(z+1) = -g(z)$, $g(0) = 0$ and

$$|g(z)| \leq e^{\pi|\,\mathrm{Im}(z)\,|} .$$

   Show that there exists a constant $b \in \mathbb{C}$ such that

$$g(z) = b \sin\left(\pi\, z\right) .$$

3. (8+10+7 = 25 pt.) Let $\Omega \subset \mathbb{C}$ be a domain and $f(z)$ be a meromorphic function in Ω with a non-empty set W of poles. Choose an arbitrary point $z_0 \in \Omega - W$.

   a) Show that W is a discrete subset of Ω.

   Give an example where Ω is bounded and W is an infinite set.

   b) Show that if the residue of $f(z)$ at each pole vanishes, then

- for $z \in \Omega - W$ the integral

$$F(z) = \int_{z_0}^{z} f(u)du$$

is independent of the path $\Gamma \subset \Omega - W$ connecting z_0 and z, and

- $F(z)$ defines an analytic function in $\Omega - W$.

c) True or false ? Explain.

$F(z)$ is meromorphic in Ω with W as the set of poles.

4. (10+8+7 = 25 pt.) Let $g(z)$ be a non-constant entire periodic function, $f(z)$ be a meromorphic function in $\mathbb{C}$.

a) Let z_0 be a pole of $f(z)$. Show that the function $g \circ f$ has an essential singularity at z_0 (that is, $lim_{z \to z_0}(g \circ f(z))$ does not exist).

b) For $g(z) = e^z$, prove the result in (a) by using the argument principle.

c) Show that if $f(z)$ has at least two poles then $f \circ g$ has infinitely many poles.

METU - Mathematics Department
Graduate Preliminary Exam

Complex Analysis

Duration : 3 hours **September 2006**

1. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by
$$f(x,y) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0). \end{cases}$$
Show that

   a) $f(x,y)$ is a continuous function of the variables x and y.

   b) The functions $u(x,y) = Re(f(x,y))$ and $v(x,y) = Im(f(x,y))$ satisfy the Cauchy-Riemann equations at $z = 0$.

   c) $f'(z)$ does not exist at $z = 0$.

   Why does the conclusion in (c) not contradict part (b) ?

2. Let $f(z)$ be a continuous function on the unit circle $S^1 = \{z : |z| = 1\}$. Show that the function
$$F(z) = \int_{S^1} \frac{f(\xi)}{(\xi - z)} d\xi$$
is analytic on $D = \{z : |z| < 1\}$ and that
$$F'(z) = \int_{S^1} \frac{f(\xi)}{(\xi - z)^2} d\xi.$$

3. Explain how to choose a branch of $f(z) = \sqrt{z^2 - 1}$ which is analytic in $\mathbb{C} - [-1, 1]$.

   a) Using this branch and the residue at infinity, compute $\displaystyle\int_\Gamma f(z)dz$ where Γ is the circle $|z| = \rho > 1$.

   b) Compute the improper integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2-1}}$ using the complex integral $\displaystyle\int_\Gamma \frac{dz}{f(z)}$ where Γ is given as

   (Hints :
   1) Residue at infinity is defined by res$(g(z), \infty)$ = -res$(g(1/t)/t^2, 0)$.
   2) The binomial series is given by $(1+z)^\alpha = \sum c_n z^n$ where $c_n = \dfrac{\alpha.(\alpha-1)...(\alpha-n+1)}{n!}$.)

4. Let $f(z)$ be an analytic function in the unit disk $D = \{z : |z| < 1\}$ and suppose that $|f(z)| \leq 1$ in D. Prove that if $f(z)$ has at least two fixed points, then $f(z) = z$ for all $z \in D$.

(Hint : Using a suitable automorphism of the disk reduce to the case where one of the fixed points is $0 \in D$ so that $g(z) = f(z)/z$ defines an analytic function on D.)

METU - Mathematics Department
Graduate Preliminary Exam-Fall 2007

Complex Analysis

1. Evaluate $\displaystyle\int_0^\infty e^{-x^2}\cos(x^2)dx$ using complex integration along the given contour.
   (Hint: $\displaystyle\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$).

2. Let $f : \mathbb{C}^* \to \mathbb{C}$ be an analytic map such that for all $z \in \mathbb{C}$ the set $f^{-1}(z)$ is finite (if not empty). Show that

   (i) $z = 0$ is not an essential singularity of f.

   (ii) If f is bounded in some deleted neighborhood of 0, then f is a polynomial.

3. Recall that the analytic automorphisms of the unit disk D are given by linear fractional transformations of the form $z \mapsto e^{i\theta}\dfrac{z-\alpha}{1-\overline{\alpha}z}$ for some $\theta \in [0, 2\pi)$ and $\alpha \in$ D.

   a) Using this fact prove that the analytic automorphisms of the upper half-plane $\mathcal{H}$ are given by (special) linear fractional transformations.

   b) Show that the map $\Omega = \{z : 0 < \arg(\mathrm{z}) < \frac{\pi}{2}\} \to \mathcal{H},\ z \mapsto z^2$ is an analytic isomorphism.

   Deduce that if $g \in Aut(\Omega)$, then there exists a linear fractional transformation T such that $g(z) = \sqrt{T(z^2)}$ for a suitable branch of the square root function (which branch ?).

   c) Show that there exists no linear fractional transformation which maps Ω isomorphically onto D.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function such that $|f(z)| = 1$ if $|z| = 1$. Prove that there exist $c \in \mathbb{C},\ c \neq 0$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}, |\alpha_i| \neq 0, 1$ and $m \in \mathbb{Z}$ such that

$$f(z) = cz^m \prod_1^n \frac{z-\alpha_i}{1-\overline{\alpha}_i z}.$$

METU - Department of Mathematics
Graduate Preliminary Exam

Complex Analysis

Duration : 180 min. Fall 2008

Notation : $D = \{z : |z| < 1\}$, $D^*(0; r) = \{z : |z| < r,\ z \neq 0\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$.

1. Let $\Omega \subset \mathbb{C}$ be an open connected region and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function.

   a) Show that $\det(df(z)) > 0$ for all $z \in \Omega$, except possibly for z in a discrete set $S \subset \Omega$.

   Here df is the usual differential of $f(z) = u(x, y) + iv(x, y)$ considered as a function of two variables.

   b) Show that if $z \in \Omega - S$, then there exists a neighborhood U of z in Ω such that $f|_U$ has an analytic inverse.

   b) Can you find a **bounded** region Ω and $f(z)$ analytic in Ω for which the set S is infinite ?

   Give an example, or prove that there exists no such pair (Ω, f).

2. Let $f : D^*(0; R) \to \mathbb{C}$ be a non-constant analytic function and for $a \in \mathbb{C}$ let $S_a = f^{-1}(a) \cap D^*(0; \mathbf{R/2})$.

   a) Show that there exists no such f for which the set S_a is infinite for exactly one value of a and is finite or empty for all other $a \in \mathbb{C}$.

   b) Give an example of $f : D^*(0; R) \to \mathbb{C}$ for which the set S_a is infinite for all $a \in \mathbb{C}$, except precisely for one value of a.

3. Consider the mapping $w : \mathbb{C}^* \to \mathbb{C},\ w(z) = z + 1/z$.

   a) Show that

   (i) $w(z)$ is conformal in $\mathbb{C}^*$ except at $z = \pm 1$.

   (ii) w maps the boundary of the semi-disk $U = \{z :\ |z| < 1,\ \mathrm{Im}(z) > 0\}$ onto the real axis.

   (iii) $w|_U : U \to \mathbb{C}$ is an analytic isomorphism onto $\mathcal{H}_- = \{w : \mathrm{Im}(w) < 0\}$.

b) Write the linear fractional transformation $T : \mathcal{H}_- \to D$ which satisfies the conditions $T(-3i/2) = 0, \ T(0) = i$.

(Hint : Can you determine $T^{-1}(\infty)$?).

c) Show that we obtain an analytic isomorphism $\Phi = T \circ w : U \to D$ such that $\Phi(i/2) = 0, \ \Phi'(i/2) > 0$ and that Φ is the unique isomorphism satisfying these conditions.

4. Using complex integration on the given contour Γ compute

$$\int_0^\infty \frac{dx}{x^a(1+x)}, \quad 0 < a < 1.$$

NOTE : You must specify the complex function you are integrating and justify the details of the computation.

TMS

Fall 2009

Complex Analysis

I. a) Find the conformal map

$$T : \mathcal{H} = \{z : Im z > 0\} \to D = \{z : |z| < 1\}$$

which satisfies

$T(i) = 0$

$T(1) = 1$

(Hint: Consider symmetry).

b) Map the region

$$\Omega = \{z : 0 < Re(z) < 2\}$$

conformally onto D.

(Hint: First map Ω into $\mathcal{H}$).

2. a) Formulate **precisely**, the Cauchy theorem for complex integration and its partial converse (the Morera's Theorem)

b) Using Morera's Theorem, prove that every function f which is continuous in the open disk D and analytic on $D - \{1/2\}$ is analytic on D.

3. Let f be a meromorphic function in $\mathbb{C}$ whose poles all lie on the line $y = x$ (for example $f(x) = \tan(\frac{z}{1+i}))$ and for $r \in \mathbb{R}_+$, let $C(r)$ be the circle $|z| = r$.

a) For which circles $C(r)$, is $\int_{C(r)} f(z)dz$ defined?

b) Show that the formula

$$F(r) = \int_{C(r)} f(z)dz$$

defines a function $(0, \infty) - D \to \mathbb{C}$ where D is a discrete subset of $(0, \infty)$.

c) Show that if $f(z)$ has only finitely many poles in $\mathbb{C}$, then there exists some $R > 0$ such that F defines a constant function on (R, ∞).

4. Consider the open disc $D(0,1)$. Let $a, b \in D(0,1)$ be two distinct points.

a) Write the most general analytic automorphism

$\sigma : D(0,1) \to D(0,1)$

such that $\sigma(a) = 0$.

b) Show that $\mathrm{Aut}(D(0,1))$ acts transitively on $D(0,1)$, by writing down

$$\tau \in AutD((0,1))$$

such that $\tau(a) = b$.

c) True or false? why?

If $\Omega \subset \mathbb{C}$ is any **simply connected** region, and if $a, b \in \Omega$, then there exists an analytic automorphism $\psi : \Omega \to \Omega$ such that $\psi(a) = b$.

METU - Department of Mathematics
Graduate Preliminary Exam

Complex Analysis

Duration : 180 min. Fall 2010

1. (25 pt.) Consider the following Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n} + \sum_{n=0}^{\infty} a_n z^n.$$

Suppose that $f(z)$ is analytic in the annulus $\Omega = \{z : 0 < |z| < 3\}$ and nowhere else and that $f(z)$ has a pole of order two at $z = 0$ with $Res(f; 0) = 1$.

a) Determine the coefficients $b_n,\ n \geq 1$.

b) Determine

$$lim_{n\to\infty}|a_n|^{1/n}.$$

c) Evaluate

$$\int_\gamma z^m f(z)dz$$

as $m \in \mathbb{Z}$ varies, where γ is the square whose vertices are at $1 \pm i, -1 \pm i$ oriented counter-clockwise.

d) Can you choose the coefficients so that $f(z)$ vanishes on a sequence $\{z_n : n \geq 1\}$ which converges to $z = 0$? Explain.

2. (25 pt.) For any open set $\Omega \subset \mathbb{C}$ and any discrete set $S \subset \Omega$ consider the set $\mathcal{F}_\Omega(S)$ of functions analytic on Ω which vanish at each $z \in S$ to a given order $n(z)$ and nowhere else.

a) Prove that if $f, g \in \mathcal{F}_\mathbb{C}(S)$, then there exists an entire function $h(z)$ such that $f(z) = e^{h(z)}g(z)$.

b) Suppose that S is a non-empty finite set and that $f \in \mathcal{F}_\mathbb{C}(S)$. Discuss the behaviour of the function $1/f(z)$ at infinity.

c) True or false ? Why ?

The statement in (a) is valid for all simply connected regions Ω and all discrete sets $S \subset \Omega$.

3. (25 pt.) a) Let $f : \Omega_{\text{op}} \subset \mathbb{C} \to \mathbb{C}$ be a meromorphic function with an isolated singularity at $z = a \in \Omega$. Prove that $z = a$ is a simple pole if and only if there exists an open neighbourhood $\Omega' \subset \Omega$ of $z = a$ and an analytic function $g : \Omega' \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = \frac{g(z)}{z-a}$$

for all $z \in \Omega'$. What is the residue of $f(z)$ at $z = a$?

b) Suppose that

$$\Lambda = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

and $\overline{T} : \Lambda \to \mathbb{C}$ is an analytic nowhere vanishing function such that

$$\overline{T}(z+1) = z\overline{T}(z)$$

for all $z \in \Lambda$. Prove that there exists a unique meromorphic function $T : \mathbb{C} \to \mathbb{C}$ with simple poles at $z = -n,\ n = 0, 1, 2, \ldots$ such that $T|_\Lambda = \overline{T}$.

c) Prove that

$$Res_{z=-n}(T(z)) = \frac{(-1)^n}{n!} T(1).$$

4. (25 pt.) a) Given $\Omega_{\text{op}} \subset \mathbb{C}$ and analytic $g : \Omega \to \mathbb{C}$, prove that for any $a \in \Omega$ the function $\Psi : \Omega \to \mathbb{C}$ defined by

$$\Psi(z) = \begin{cases} \dfrac{g(z) - g(a) - (z-a)g'(a)}{(z-a)^2} & \text{for } z \neq a \\ \frac{1}{2} g''(a) & \text{for } z = a \end{cases}$$

is analytic on Ω.

b) Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and consider an analytic function $f : \Delta \to \mathbb{C}$ which satisfies $f(0) = f'(0) = 0$ and $|f(z) \ \leq 1$ for all $z \in \Delta$. Prove that

$$|f(z)| \leq |z|^2$$

for all $z \in \Delta$ and $|f''(0)| \leq 2$.

(Hint : Consider the function constructed from f in the manner of (a) and then apply the maximum principle as in the demonstration of Schwarz's Lemma.)

c) Prove that in order for one of the inequalities in (b) to reduce to an equality even for a <u>s</u>ingle $z \in \Delta$, it is necessary and sufficient that $f(z) = cz^2$ for some $c \in \mathbb{C}$ with $|c| = 1$.

M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

COMPLEX ANALYSIS

Duration : 180 min.

Each question is 25 pt.

1. For $R > 0$, let Γ_R be the counterclockwise oriented closed curve obtained by joining the following paths, in the given order :

   $\Gamma_{1,R} : [0, R] \to \mathbb{C}$ defined by $\Gamma_{1,R}(t) = t$
   $\Gamma_{2,R} : [0, \pi/2] \to \mathbb{C}$ defined by $\Gamma_{2,R}(t) = Re^{ti}$
   $\Gamma_{3,R} : [0, R] \to \mathbb{C}$ defined by $\Gamma_{3,R}(t) = (R - t)i$.

   Consider the analytic function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ defined by

   $$f(z) = \frac{e^{-z} - e^{iz}}{z}$$

   for each $z \neq 0$.

   (A) Show that $z = 0$ is a removable singularity of $f(z)$.

   (B) Prove that

   $$\int_{\Gamma_R} f(z)dz = 0 \ .$$

   (C) Prove that

   $$\lim_{R\to\infty} \int_{\Gamma_{2,R}} f(z)dz = 0 \ .$$

   (D) Prove that

   $$\int_0^\infty \frac{e^{-x} - \cos x}{x} dx = 0 \ .$$

2. Consider

$$\Omega = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

and

$$\Delta = \{z \in \mathbb{C} \mid |\, z \,| < 1\}\ .$$

(A) Given any $b \in \Omega$ prove that the map $\Psi = \Psi_b : \mathbb{C} - \{\bar{b}\} \to \mathbb{C}$ defined for each $z \neq \bar{b}$ by

$$\Psi(z) = \frac{z-b}{z-\bar{b}}$$

maps Ω onto Δ bijectively.

(B) If $f : \Omega \to \mathbb{C}$ is analytic and satisfies $f(\Omega) \subseteq \Omega$, prove that

$$\frac{|f(z) - f(a)|}{|f(z) - \overline{f(a)}|} \leq \frac{|z-a|}{|z-\bar{a}|}$$

for any $z, a \in \Omega, z \neq a$. [1]

(C) Deduce that

$$|f^{'}(z)| \leq |\frac{\mathrm{Im}(f(z))}{\mathrm{Im}(z)}|$$

for any $z \in \Omega$.

[1] Consider $g = \Psi_{f(a)} \circ f \circ \Psi_a^{-1}$.

3. Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be an open connected region and let

$$f = (u, v) : \Omega \to \mathbb{C}$$

be a **non-constant differentiable** function. Consider the set

$$Z_{df} = \{z = (x, y) \in \Omega : det(df(x, y)) = 0\}.$$

a) Give an example $f(x, y)$ for which Z_{df} is not a discrete subset of Ω.

b) Suppose that $f(z)$ is an analytic function.

- Show that Z_{df} is discrete.
- Suppose that $z_0 \in Z_{df}$ and that $f(z_0) = 0$. Show that there exists an integer $n > 1$ such that the function $g(z) = 1/f(z)$ maps a neighborhood of z_0 analytically onto a neighborhood of ∞ in an n-to-one manner.
- Show that if $\text{Res}(g; z_0) = 0$ then $g(z)$ is the derivative of a function meromorphic around z_0.

4. a) Show that if $f(z)$ is a non-constant entire periodic function, then $f(z)$ has an essential singularity at ∞.

b) Does there exist a non-constant entire doubly periodic function ? Explain.

(Recall that a meromorphic function $f(z)$ is said to be doubly periodic with periods w_1, w_2 if $w_1/w_2 \notin \mathbb{R}$ and $f(z + w_1) = f(z) = f(z + w_2)$ for all $z \in \mathbb{C}$).

c) Let $f(z)$ be a doubly periodic function with periods w_1, w_2. Let $a \in \mathbb{C}$ be such that $f(z)$ has no poles on the boundary of the parellogram

Γ whose vertices are at $a,\ a + w_1,\ a + w_2,\ a + w_1 + w_2$.

Show that $f(z)$ has finitely many poles $\{z_1, ..., z_N\}$ in the interor of Γ and that

$$\sum_1^N \text{Res}(f; z_i) = 0.$$

PRELIMINARY EXAM - Sep.2012
Complex Analysis

Duration : 3 hr.

Q.1	Q.2	Q.3	Q.4	Total

1. (25 = 20+5 pt.)

   a) Let

   $$f : \overline{D}(0;1) \to \mathbb{C} \quad \text{and} \quad g : \overline{D}(0;1) \to \mathbb{C}$$

   be two continuous functions which are analytic on $D(0;1)$. Show that if $f = g$ on the unit circle $|z| = 1$, then $f = g$.

   b) Give an example of a pair of smooth functions f, g on $\overline{D}(0;1)$ such that $f \neq g$ on $D(0;1)$ but $f(x,y) = g(x,y)$ on the circle $x^2 + y^2 = 1$.

2. (25 = 7+10+8 pt.)

   Let $\Omega \subset \mathbb{C}$ be an open region and let $f(z)$ be a function which is analytic on Ω except for a set of isolated singularities.

   a) Show that Residue$(f'(z); z_0) = 0$ for all $z_0 \in \Omega$.

   b) Show that if $f(z)$ is meromorphic on Ω, then the function

   $$g(z) = e^{f(z)}$$

   has no poles in Ω.
   (Hint : For $z_0 \in \Omega$, apply the principle of argument to $g(z)$ in a suitable neighbourhood of z_0).

   c) Construct an analytic function $h : \mathbb{C} - \{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}$ which has an essential singularity at each point $z_n = n\pi, n \in \mathbb{Z}$.

3. $(25 = (7+3) + 7 + 8$ pt.)

$\mathsf{n}(\gamma, \mathsf{a})$ denotes the *index* at $a \in \mathbb{C}$ of the closed curve $\gamma : [0, 2\pi] \to \mathbb{C} - \{a\}$.

(A) If $b \neq 0$, prove that

$$\mathsf{n}(\gamma^n, b^n) = \mathsf{n}(\gamma, b).$$

Prove also that

$$\mathsf{n}(\gamma^n, 0) = n\,\mathsf{n}(\gamma, 0)$$

for $n \in \mathbb{Z}$ with $n \geq 1$.

(B) If $a \in \mathbb{C} - \{0\}$ is an isolated singularity of analytic $f(z)$, prove that any b with $a = b^n$ is an isolated singularity of $g(z) = f(z^n)$.

(C) Show that $\frac{\mathrm{Res}_{z=a}(f(z))}{a} = n \frac{\mathrm{Res}_{z=b}(g(z))}{b}$

4. $(25 = 5 + 8 + 7 + 5$ pt.)

Consider an open $\Omega \subseteq \mathbb{C}$ and $a \in \Omega$. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

(A) If $h : \Omega \to \mathbb{C}$ is a continuous function which is analytic on $\Omega - \{a\}$, prove that h is analytic on Ω.

(B) Let $f : \mathbb{C} - \Delta \to \Delta$ be an analytic function with $\lim_{z\to\infty} f(z) = 0$.
Prove that

$$|f(z)| \leq \frac{1}{|z|}$$

for all $z \in \mathbb{C} - \Delta$.

(C) Prove that $\lim_{z\to\infty}\Big(zf(z)\Big)$ exists and $\Big|\lim_{z\to\infty}\Big(zf(z)\Big)\Big| \leq 1$.

(D) Prove that in order for any one of the inequalities in (B) and (C) to become an equality, it is necessary and sufficient that f is a constant.

Complex Analysis

PRELIMINARY EXAMINATION

Monday, 23rd September 2013
Four questions, three hours

1

[4 + 6 + 7 + 8]

(A) Let f be an entire function. If $\text{Re}(f)$ is bounded, prove that f is a constant. (Consider e^f !)

(B) Let g be an entire function. If $a\,\text{Re}(g) - b\,\text{Im}(g)$ is bounded, where $(a,b) \neq (0,0)$, prove that g is a constant.

(C) Show that for a not identically vanishing entire function h, the quantity $x\,\text{Re}(h(z)) - y\,\text{Im}(h(z))$ is unbounded, where $z = x + iy$.

(D) Does there exist an entire function w with $w'(0) = 1$ such that

$$x\,\text{Re}(w(z)) + y\,\text{Im}(w(z)) \leq x^2 + y^2$$

for all $z = x + iy \in \mathbb{C}$.

2

[12 + 13]

(A) Show that

$$\exp\left(\frac{z - z^{-1}}{2}\right) = \sum_{n=-\infty}^{\infty} A_n z^n$$

for every $z \in \mathbb{C} - \Big\{0\Big\}$ where,

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} \cos\Big(n\theta - \sin\theta\Big) \quad .$$

(B) Show - by employing the Taylor expansion of e^w about $w = 0$ or otherwise - that

$$\int_0^{2\pi} \cos\Big(\theta + \sin\theta\Big) = \pi \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k\, k!\,(k+1)!} \quad .$$

$\boxed{3}$

[5 + 7 + (4 + 9)]

(A) Let $f : \Omega \subseteq_{\text{op}} \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function. If $a \in \Omega$ is a zero of f with multiplicity $\mu \geq 2$, prove that $f'(a) = 0$.

(B) Let $g : \Omega \subseteq_{\text{op}} \mathbb{C} \longrightarrow \mathbb{C}$ be another analtytic function, $a_1, a_2, a_3, \cdots, a_N \in \Omega$ be the zeros of f with respective multiplicities $\mu_1, \mu_2, \mu_3, \cdots, \mu_N$ enclosed by the simple closed positively oriented curve Γ. Prove that

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(z)\, f'(z)}{f(z)}\, dz = \sum_{n=1}^{N} g(a_n)\mu_n$$

(C) Prove that the polynomial

$$\mathrm{e}(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

has exactly n distinct rooots $b_1, b_2, b_3, \cdots, b_n \in \mathbb{C}$ and

$$\sum_{r=1}^{n} b_r^{-t} = 0$$

for every $t \in \mathbb{N}$ with $2 \leq t \leq n$.

$\boxed{4}$

[7 + 9 + 9]

(A) Let

$$\Omega = \left\{ z \in \mathbb{C} \;\middle|\; |z| > 4 \right\} .$$

Show that there exists an analytic function $f : \Omega \longrightarrow \mathbb{C}$ with

$$f'(z) = \frac{z}{(z-1)(z-2)(z-3)} .$$

(B) Show that there exists no analytic function $g : \Omega \longrightarrow \mathbb{C}$ with

$$g'(z) = \frac{z^2}{(z-1)(z-2)(z-3)} .$$

(C) Let

$$\Xi = \left\{ z \in \mathbb{C} \;\middle|\; |z| > 9/4\,,\ |z-3| > 1/8 \right\} .$$

Does that there exists an analytic function $h : \Xi \longrightarrow \mathbb{C}$ with

$$h'(z) = \frac{z}{(z-1)(z-2)(z-3)} \qquad ?$$

Complex Analysis

PRELIMINARY EXAMINATION

Monday, 15th September 2014. Four questions, three hours.

1

[12 + 7 + 6]

(A) Prove that the real and imaginary parts of $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \exp\left(-\dfrac{1}{z^4}\right) & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

have partial derivatives that satisfy the Cauchy-Riemann equations at any $z \in \mathbb{C}$.

(B) Is $f(z)$ differentiable as a function of z at $z = 0$?

(C) Give conditions under which the Cauchy-Riemann equations are sufficient for differentiability with respect to z. Indicate why they are not applicable in this case.

2

[(7 + 7) + 11]

(A) Given an entire function f evaluate

$$\int_{\Gamma_R} \frac{f(z)}{(z-a)(z-b)}\, dz$$

for any $a, b \in \mathbb{C}$ where $R > |a|, |b|$ and the contour Γ_R is the counterclockwise traversed circle of center $0 \in \mathbb{C}$ and radius R. Use this relation to prove the Liouville theorem to the effect that a bounded entire function reduces to a constant.

(B) Let h be an entire function that h has simple zeros, only. If g is an entire function which satisfies

$$|g(z)| \leq |h(z)|$$

for all $z \in \mathbb{C}$, prove that $g(z) = ch(z)$ for some constant $c \in \mathbb{C}$ with $|c| \leq 1$.

3

[15 + 10]

(A) Prove that the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where $a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}$ with

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$$

has no roots in the open disc

$$\Delta = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\} \quad .$$

(*Hint* : Consider the polynomials $(z-1)p(z)$ and $a_n z^{n+1} - a_0$ on the boundary of Δ .)

(B) Prove that the polynomial

$$q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n$$

where $b_0, b_1, b_2, \cdots, b_n \in \mathbb{R}$ with

$$b_n \geq \cdots \geq b_1 \geq b_0 > 0$$

has exactly n roots (counted with multiplicities) in the closure of Δ .

4

[8 + 17]

Use your own choice of branches of the squareroot and the logarithmic function to prove by calculus of residues, or otherwise, that

$$\int_0^\infty \frac{\sqrt{x}\log(x)}{x^2+1}\, dx = \frac{\pi^2}{2\sqrt{2}} \quad .$$

Preliminary Exam - February, 2015
COMPLEX ANALYSIS

Each question is 25 pt.

1. a) Show that the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ \ f(x,y) = (\sin(2x+y), \cos(2x+y))$$

is differentiable everywhere.

Determine all points, if any, where f has a local differentiable inverse.

b) Show that as a complex valued function of the complex variable $z = x+iy$, $f(z)$ is nowhere analytic by

(i) checking the Cauchy-Riemann conditions,

(ii) using the topological mapping defined by f.

c) Can you find an entire function $g(z)$ such that $\mathrm{Re}(g(z)) = \sin(2x+y)$? Explain.

2. Let $f,\ g$ be two entire functions which have simple zeros precisely at

$$z_n = \frac{(2n+1)}{2}\pi,\ \ n \in \mathbb{Z}$$

and no other zeros.

a) Show that the function $\dfrac{f}{g}(z)$ is entire.

b) Show that $\dfrac{f}{g}(z) = e^{h(z)}$ for some entire function $h(z)$.

c) For $N \geq 1$, let $a_N = \displaystyle\int_\gamma \frac{f'(z)}{f(z)} dz$ where γ is the circle $|z| = 1 + N\pi$.

True or false ? Explain.

The series $\sum_1^\infty \dfrac{1}{a_N^{1+r}}$ converges for all $r > 0$.

3. a) Show that if $\Omega \subset \mathbb{C}$ is a domain such that $\mathrm{Aut}(\Omega)$ is a finite group, then Ω is not simply connected.

b) True or false ? Give an example or disprove the statement.

There exists an analytic function $f : \mathbb{C} - 0 \to \mathbb{C}$ such that $f(\mathbb{C} - 0) = \mathbb{C}$ and f has an essential singularity at $z = 0$.

c) Show that the family of functions $f_n(z) = z^n, n \geq 1$ is normal in $D(0; 1)$, but not in any region which contains a point on the unit circle.

4. Let $S = \{z_n \in \mathbb{C} : n \geq 1\}$ be a discrete set and let

$$f : \mathbb{C} - S \to \mathbb{C}$$

be a holomorphic function.

a) Suppose that $\mathrm{Res}(f; z_1) = 0$. Show that there exist $R > 0$ and a holomorphic function $g(z)$ in $D^*(z_1; R)$ such that $f(z) = \dfrac{dg}{dz}(z)$.

b) Show that $f(z) = \dfrac{dh}{dz}(z)$ for some analytic function $h(z)$ in $\mathbb{C} - S$ if and only if $\mathrm{Res}(f; z_n) = 0$, for all $n \geq 1$.

Preliminary Exam - September 2015
COMPLEX ANALYSIS

Each question is 25 pt.

1. Consider the entire function $f(z) = z^2 + Bz$.

   a) Determine B if $|f(B)| = |B|$ and at $z = 0$ the given map defines a rotation through $\theta = \pi/4$.

   b) Determine the zeros, poles and all multiple points of the holomorphic map

   $$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

   c) Let $g(z) = f(z)^{-n}$, $n \geq 1$. Determine

   $$\int_{|z|=R} \frac{g'(z)}{g(z)} dz$$

   for all permissible values of R.

2. Consider the function $f(z) = e^{1/z(z-1)}$.

   a) Show that f is holomorphic in $\mathbb{C} - \{0,\ 1\}$ with essential singularities at $z = 0$ and $z = 1$.

   b) Compute

   $$\int_{z=R} f(z) dz$$

   for

   (i) $0 < R < 1$, (ii) $R > 1$.

   c) True or false ? Why ?

   For any pair of positive real numbers $\{r,\ r'\}$, $f(D^*(0; r)) \cap f(D^*(1; r')) \neq \emptyset$.

3. Let $\sum_0^\infty a_n z^n$ be a power series which converges in $\mathbb{C}$ to a nowhere vanishing function $f(z)$.

a) Show that for any given $R > 0$, there exists a positive integer N such that for all $m \geq N$, the polynomial $f_m(z) = \sum_0^m a_n z^n$ has no zeros in $D(0; R)$.

b) True or false ? Explain.

(i) $log(f(z))$ can be defined as an entire function.

(ii) f extends to a meromorphic function on the extended complex plane if and only if it is constant.

4. a) Show that if $f : \overline{D}(0; R) \to \mathbb{C}$ is holomorphic and $|f(z)| < R$ on $|z| = R$, then there exists a unique point $a \in D(0; R)$ such that $f(a) = a$.

b) Show that for $a,\ b \in \Omega$ (Ω a simply connected region), there exists a holomorphic automorphism

$$\Phi : \Omega \to \Omega$$

such that $\Phi(a) = b$. Is Φ unique ?