20 February 2004

Graduate Preliminary Examination

Geometry

Duration: 3 hours

1. Let S^2 be the unit circle in $\mathbb{R}^3$. Considering S^2 oriented by outer normal field

a) exhibit a positively oriented basis of the tangent space for each point of S^2,

b) determine whether the reflection $F : S^2 \to S^2$ which is given by $F(x, y, z) = (x, -y, z)$ is orientation preserving or not.

2. Let X, Y be smooth vector fields on a smooth manifold M. Then XY defined by $(XY)(f) = X(Yf)$ makes sense as a smooth operator. We know that $[X, Y] = XY - YX$ is a smooth vector field.

a) Show that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for all smooth real valued functions f and g on M.

b) Let $(U; x_1, \cdots, x_n)$ be a coordinate neighborhod on M and let $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ be the associated coordinate frames. Show that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for each i, j with $1 \leq i \leq n$, $1 \leq j \leq n$.

c) Assuming that dim $M = 2$, compute the components of $[X, Y]$ in terms of the components of X and Y with respect to a coordinate neighborhood.

3. Let $F : M \to N$ be a smooth map, $q \in N$ a regular value and $L = F^{-1}(q) \subset M$. Show that for any $p \in L$ the tangent space T_pL is the kernel of the induced map $F_* : T_pM \to T_qN$.

4. Let w be the 2-form on $\mathbb{R}^3 \setminus (0, 0, 0)$ given by $w = d(\frac{1}{x^2+y^2+z^2}dy)$.

a) Find the local expression of the pull back of w on M with respect to the local parametrization

$$
\begin{aligned}
x &= 2\cos\ u\ \ (1+\cos v) - 2 \\
y &= 2\sin\ u\ \ (1+\cos v) \\
z &= \sin v \qquad\qquad\qquad u, v \in (0, 2\pi).
\end{aligned}
$$

b) Find $\displaystyle\int_M w$.

METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Geometry

Duration: 3 hours

February 18, 2005

1. Consider the set $M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1\ ,\ z^2 + w^2 = 1\} \subseteq \mathbb{R}^4$.

- (**a**) Prove that M is an (imbedded) submanifold of $\mathbb{R}^4$.
- (**b**) Describe the tangent vectors of M at an arbitrary point $(a, b, c, d) \in M$.
- (**c**) Write down a nowhere vanishing vector field on M .
- (**d**) Let $\omega = (ydx - xdy) \wedge (wdz - zdw) \in \Omega(\mathbb{R}^4)$. Show that $\int_M i_\star(w) > 0$ where $i : M \to \mathbb{R}^4$ is the inclusion map (Hint: Write a local parametrization for M).
- (**e**) A consequence of Poincaré Lemma is that every closed form on $\mathbb{R}^n$ for any n is also exact. Prove that there exists no 4-form $\theta \in \Omega(\mathbb{R}^4)$ with $d\theta = 0$ such that $\int_M i^\star(\theta) \neq 0$.

2. Consider the the $(k-1)$ dimensional sphere S^{k-1} as a submanifold of S^k via the usual embeddding $(x_1, x_2, \ldots, x_k) \to (x_1, x_2, \ldots, x_k, 0)$. Show that the orthogonal complement to $T_p(S^{k-1})$ in $T_p(S^k)$ is spanned by the vector $(0, 0, \ldots, 1)$.

3. Let ω be a compactly supported 2-form

$w = f_1\ dx_2 \wedge dx_3 + f_2\ dx_3 \wedge dx_1 + f_3\ dx_1 \wedge dx_2$

on $\mathbb{R}^3$. Let S be the graph of a function $G : \mathbb{R}^2 \to \mathbb{R}$. Compute the integral $\int_S \omega$, and show that it is equal to $\int_{\mathbb{R}^2} (\vec{F}.\vec{u})\ \|\vec{n}\|\ dx_1 \wedge dx_2$ where $\vec{F} = (f_1, f_2, f_3)$, $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ with $\vec{n} = (-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1)$.

4. Consider the sets

$M_1 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 + v^2 = w^2\} \subseteq \mathbb{R}P^2$.

$M_2 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 - v^2 = w^2\} \subseteq \mathbb{R}P^2$.

(**a**) Prove that M_1 is an (imbedded) submanifold of $\mathbb{R}P^2$ diffeomorphic to $\mathbf{S}^1$ (Hint: Consider the image of M_1 under a suitable chart of $\mathbb{R}P^2$).

(**b**) Find a diffeomorphism $F : \mathbb{R}P^2 \to \mathbb{R}P^2$ such that $F(M_1) = M_2$.

METU - Mathematics Department
Graduate Preliminary Exam-Fall 2010

Geometry

1.a. Let $v_1 = (2, -3, -1)$, $v_2 = (0, 4, 8)$ and $v_3 = (2, 0, 0)$ be vectors in $\mathbb{R}^3$. Calculate $(dx \wedge dz)(v_1, v_2)$ and $(dx \wedge dy \wedge dz)(v_1, v_2, v_3)$.

1.b. Let $\omega = (-2x + y)\ dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r, s, t) = (r - t, r^2 s)$. Calculate $f^*(\omega)$, the pullback of ω by f.

1.c. Repeat Part (b) for the constant function $f(r, s, t) = (2, -5)$, for any $(r, s, t) \in \mathbb{R}^3$.

2.a. Let ω be the 2-form on $\mathbb{R}^3 - \{(0, 0, 0)\}$ given by

$$\omega = \frac{1}{4\pi} \frac{x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that ω is closed.

2.b. Calculate the integral of ω over the 2-torus shown in the figure below. What would your answer be if the origin were inside the 2-torus?

3.a. Show that the smooth map $\Phi : S^2 \to \mathbb{R}^5$, given by $\Phi(x, y, z) = (x^2, y^2, xy, xz, yz)$ is an immersion, where S^2 is the unit sphere in $\mathbb{R}^3$.

3.b. Show that Φ is a 2-to-1 map with $\Phi(x, y, z) = \Phi(-x, -y-, z)$. Conclude that Φ gives a closed embedding of the real projective plane $\mathbb{R}P^2 = S^2/\sim$, where the equivalence relation $\sim$ on S^2 is defined by, for $p, q \in S^2$ we have $p \sim q$ if and only if $p = -q$.

4.a. Show that 1 is a regular value of the smooth map $F : \mathbb{R}^4 \to \mathbb{R}$ given by $F(a, b, c, d) = ad - bc$. Conclude that the set of 2×2-matrices of determinant one, $SL(2, \mathbb{R})$, is a submanifold of the manifold of all 2×2-matrices $M(2, \mathbb{R}) = \mathbb{R}^4$. What is the dimension of $SL(2, \mathbb{R})$?

4.b. Is $0 \in \mathbb{R}$ a regular value of the same F? Justify your answer.

DIFFERENTIABLE MANIFOLDS, FEBRUARY 2011
TMS EXAM

FEBRUARY 18, 2011

1.a) Let $\omega = (x + yz)\ dx \wedge dy + dx \wedge dz$, a 2-form on $\mathbb{R}^3$, and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(s,t) = (t+s, 2s+e^t)$. Calculate $f^*(\omega)$, the pullback of ω by f.

1.b) Consider the vector field on the space

$$X = 2x\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}.$$

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^3 \to \mathbb{R}$.

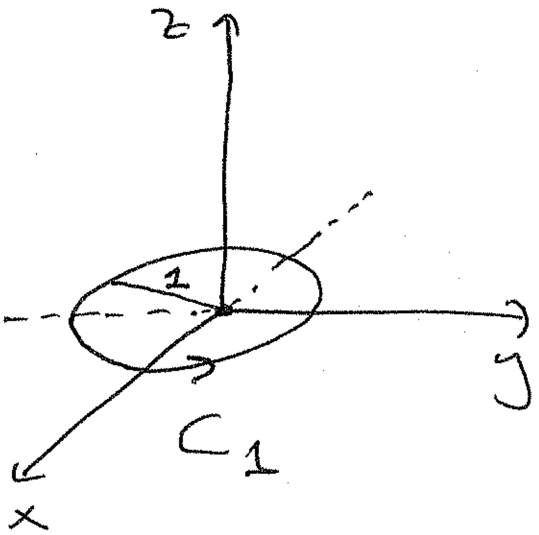
2.a) Let ω be the 1-form on $\mathbb{R}^3 - \{(x,y,z) \mid x = 0,\ y = 0\}$ given by

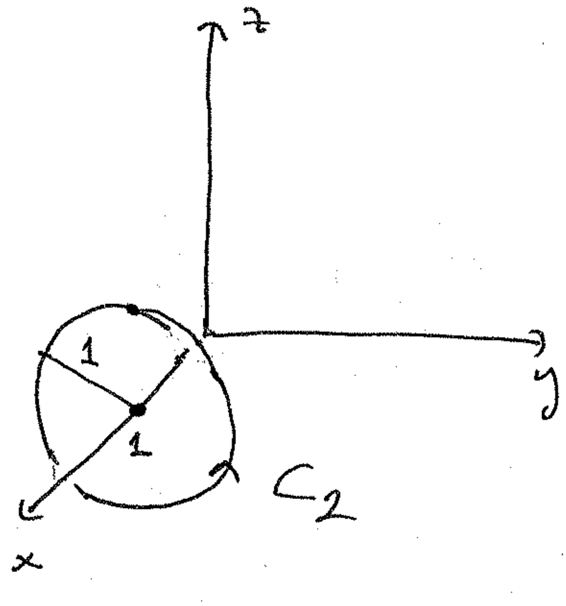
$$\omega = \frac{1}{2\pi}\frac{x\ dy - y\ dx}{x^2 + y^2}.$$

Show that ω is closed.

2.b) Calculate the integral of ω over the circles shown in the figure below.

$$C_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0,\ x^2 + y^2 = 1\}\ ,$$
$$C_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x = 1,\ y^2 + z^2 = 1\}.$$





3.a) Prove that the subset $C = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 + 1\}$ is a smooth manifold by showing that $0 \in \mathbb{R}$ is a regular value for the function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = z^2 - x^2 - y^2 - 1$. What is its dimension? Describe its tangent space at any point $(a, b, c) \in C$.

3.b) Calculate the differential the smooth map $\Phi : M(n) \to S(n)$, $\Phi(A) = A^t A$, at the identity matrix I_n, where $M(n)$ is the set of all $n \times n$ matrices over reals and $S(n)$ is the set of symmetric real matrices over reals. Is the identity matrix I_n a regular value for Φ? (Hint: Note that we may regard $M(n)$ as $\mathbb{R}^{n^2}$ and $S(n)$ as $\mathbb{R}^{n(n+1)/2}$.)

4) Consider the 2-form on $\mathbb{R}^4$ $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

a) Calculate $\omega \wedge \omega$.

b) Can we write $\omega = \nu \wedge \eta$ for some 1-forms ν and η on $\mathbb{R}^4$?

c) Show that ω is closed. Let $S \subseteq \mathbb{R}^4$ be an embedded compact connected and orientable surface without boundary. Calculate the integral $\int_S \omega$.

Differentiable Manifolds
TMS EXAM
11 February 2013

Duration: 3 hr.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1 \ .$$

For which of the points $p = (0, 0)$, $p = (1/3, 1/3)$, $p = (-1/3, -1/3)$ is $f^{-1}(f(p))$ an imbedded submanifold in $\mathbb{R}^2$?

2. Let M be the hyperboloid of two sheets given by $y^2 - z^2 - x^2 = 1$.

(**a**) Let $p \in M$. Explain how we can identify T_pM by a subspace of $\mathbb{R}^3$ using a chart at p.

(**b**) Describe $T_p(M)$ as a subspace of $\mathbb{R}^3$ if $p = (0, 2, \sqrt{3})$.

(**c**) Determine whether the map which assigns to each point $q = (x, y, z)$ the vector $(y, x+z, y)$ is a smooth vector field on M.

3. Let $F : M \to N$ be a smooth function between the manifolds M and N and let a be a smooth function on M.

(**a**) Show that $F^*(da) = d(F^*(a))$

(**b**) Verify the formula $F^*d = dF^*$ on the forms of type $\phi_1 \wedge \phi_2$ where ϕ_1 and ϕ_2 are 1-forms.

(**c**) Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$g(x, y, z) = (xy, x^2yz)$$

Compute $g^*(2xydx \wedge dy)$

4. Let

$$\alpha = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}$$

(**a**) Prove that α is a closed 1-form on $\mathbb{R}^2 \setminus 0$

(**b**) Compute the integral of α over the unit circle S^1 ?

(**c**) How does this shows that α is not exact?

Differentiable Manifolds
TMS EXAM
13 February 2015

Duration: 3 hr.

1. Let S^2 be the unit sphere in $\mathbb{R}^3$. Consider it with the topology relative to $\mathbb{R}^3$. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(**a**) Show that i is an immersion.

(**b**) Is i an embedding? Why?

2. Let M, N be two differentiable manifolds and $f : M \to N$ be a smooth map. Define a new map $F : M \to M \times N$ by $F(p) = (p, f(p))$.

(**a**) Show that F is smooth.

(**b**) Show that $F_*(v) = (v, f_*(v))$ where F_* and f_* are induced maps at a point p of M and v is a tangent vector of M at p.

(**c**) Show that the tangent space to graph(f) at the point $(p, f(p))$ is the graph of $f_* : T_pM \to T_{f(p)}N$

3. Consider the 1-form $w = (x^2 + 7y)dx + (-x + y\sin \mathrm{y}^2)\mathrm{dy}$ on $\mathbb{R}^2$.

(**a**) Is w exact? Is it closed?

(**b**) Compute the integral of w over each side of the triangle whose vertices are $(0, 0), ((1, 0), (0, 2)$ where the sides are oriented in such a way that the triangle is oriented counterclockwise.

4. Let F: $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map $F(p) = -p$.

(**a**) What is the induced map F_*? Why?

(**b**) Show that antipodal map $A : S^n \to S^n$ which is the restriction of F on the n-sphere is orientation preserving if and only if n is odd.

(**c**) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

Graduate Preliminary Examination
Differentiable Manifolds
Duration: 3 hours

September 26, 2003

1. We identify $\mathbb{R}^4$ with the set of 2×2 real matrices.

(5 pts.) (a) Show that the set $SL(2, \mathbb{R})$ of 2×2 real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?

(5 pts.) (b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, may be identified with the set of matrices of zero trace.

2. (3 pts.) (a) Show that the 1-form $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$ defined on $\mathbb{R}^2 - \{(0,0)\}$ is closed.

(3 pts.) (b) Calculate the integral $\displaystyle\int_{S^1} \omega$, where S^1 is the unit circle in $\mathbb{R}^2$.

(4 pts.) (c) Let Σ be the smooth surface shown below with boundary C. Prove that there is no smooth map $\phi : \Sigma \to S^1$ such that $\phi_{|C} : C \to S^1$, the restriction of ϕ to the boundary C, is a diffeomorphism.

3. Let $f : X \to Y$ is a smooth map between manifolds, f^* is the induced map between the algebras of differential forms of X and Y and d is the exterior derivative.

(5 pts.) (a) Prove that $d \circ f^* = f^* \circ d$.

(5 pts.) (b) If $X = \partial W$ for some compact smooth manifold W, and ω is a closed n-form on Y with $n = \dim X$, then show that

$$\int_X f^*(\omega) = 0.$$

4. (10 pts.) A curve in a manifold X is a smooth map $t \mapsto c(t)$ of an interval of $\mathbb{R}^1$ into X. The velocity vector of the curve c at time t_0 - denoted simply by $\frac{dc}{dt}(t_0)$ is defined to be the vector $dc_{t_0}(1) \in T_{x_0}X$, where $x_0 = c(t_0)$ and $dc_{t_0} : \mathbb{R}^1 \to T_{x_0}X$ is the differential of c at t_0. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \cdots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t), \cdots, c_k'(t)).$$

Prove that any vector in T_xX is the velocity vector of some curve in X, and conversely.

METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Geometry

Duration: 3 hours

September 24, 2004

1. Consider $(0,2)$-tensor field T and a $(1,1)$-tensor field S on $\mathbb{R}^2$, with the components $T_{i,j} = S^i_j = i - j + 2,\ i,j = 1,2$, where $\mathbb{R}^2$ is considered as a manifold with usual coordinates (i.e. with coordinates with respect to the standard basis e_1, e_2)

(**a**) Determine the components $T_{\alpha\beta}$ S^α_β of T and S when the coordinates in $\mathbb{R}^2$ are considered with respect to the basis $f_1 = e_1 + e_2$ and $f_2 = 2e_1 + e_2$

(**b**) Determine the components of $\operatorname{Alt} T$ and $\operatorname{Sym} T$ with respect to the basis e_1, e_2).

2. For each point $p = [u, v, w]$ on $\mathbb{R}P^2$ define curves γ_p and σ_p by

$$\begin{aligned} \gamma_p(t) &= [u, e^{-t}v, e^{-t}w] \\ \sigma_p(t) &= [u\cos t - v\sin t, u\sin t + v\cos t, w] \end{aligned}$$

for $t \in \mathbb{R}$. Consider the vector fields $A, B \in \mathfrak{X}(\mathbb{R}P^2)$ which assigns the values $\gamma_p'(0)$ and $\sigma_p'(0)$ respectively to each point $p \in \mathbb{R}P^2$

(**a**) Introduce a chart of your own choice on $\mathbb{R}P^2$ and find local expressions for A, B on this chart.

(**b**) Find local expressions for the Lie bracket $[A, B]$ on the same chart.

(**c**) For each point $p = [u, v, w]$ on $\mathbb{R}P^2$ find a curve $\theta_p : \mathbb{R} \to \mathbb{R}P^2$ such that $\theta_p(0) = p$ and $[A, B]$ takes the value $\theta_p'(0)$ at the point $p \in \mathbb{R}P^2$.

3. Consider the two dimensional sphere

$$\mathbf{S}^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} \subseteq \mathbb{R}^3$$

with its usual smooth structure and the smooth maps $f, g : \mathbf{S}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f((u,v,w)) &= w \\ g((u,v,w)) &= u \end{aligned}$$

(**a**) Evaluate the integral

$$\int_M df \wedge dg$$

where M is the manifold with boundary defined by

$$M = \{(u,v,w) \in \mathbf{S}^2 \mid v \geq 0\}$$

<u>without</u> employing Stokes' theorem.

(**b**) Use Stokes' theorem to evaluate the same integral.

4. Let M be a compact manifold and let $f : M \to N$ be a submersion where N is an arbitrary manifold with $\dim M = \dim N$. Define a function $\varphi : N \to \mathbb{R} \cup \{\infty\}$ by

$$\varphi(y) = \text{number of points in } f^{-1}(y)$$

(**a**) Prove that $\varphi(y)$ is finite for each $y \in N$.

(**b**) Prove that $\varphi : N \to \mathbb{R}$ is a locally constant function.

METU - Mathematics Department
Graduate Preliminary Exam

Geometry

Duration : 3 hours **Fall 2005**

1. **a)** Show that a one-to-one immersion of a **compact** manifold is an imbedding.

   b) Explain, in full details, why the map $\phi : (-\pi, \pi) \to \mathbb{R}^2,\ \phi(s) = (\sin(2s), \sin(s))$ shows that the conclusion in part (a) is false if X is not compact.

2. Let $SL_n(\mathbb{R})$ denote the $n \times n$ real matrices with determinant 1.

   a) Show that $SL_n(\mathbb{R})$ is a submanifold of the $n \times n$ matrices $M_n(\mathbb{R})$.

   b) Show that the tangent space to $SL_n(\mathbb{R})$ at the identity matrix I is $T_I SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : trace(A) = 0\}$.

3. **a**) What is meant by an orientation on a manifold ?

   b) Show that $S^n = \{\overline{x} \in \mathbb{R}^{n+1} : |\overline{x}| = 1\}$ is an oriented manifold, by defining an orientation on it.

   c) Show that the antipodal map $S^n \to S^n,\ \overline{x} \mapsto -\overline{x}$ is orientation preserving if and only if n is odd.

   d) Using (c), or otherwise show that $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. **a)** Show that $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a closed submanifold of $\mathbb{R}^3$.

   b) Verify that the restriction $\omega|_X$ of $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$ is a closed 1-form on X.

   c) Calculate $\displaystyle\int_S \omega|_X$, where S is the circle $\{(x, y, 3) : x^2 + y^2 = 1\} \subset X$. Is $\omega|_X$ an exact form ? Why ?

   d) Consider the mapping $\Psi : \mathbb{R}^2 \to X,\ \Psi((s, t)) = (\cos(s), \sin(s), t)$. Show that Ψ is a differentiable map and that the form $\Psi^*(\omega|_X)$ is exact.

METU - Mathematics Department
Graduate Preliminary Exam-Fall 2007

Differentiable Manifolds

1. Let $\Phi : M \to N$ be a submanifold where $\dim(M) > 1$ and let

$$\Phi^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

be the restriction map $f \mapsto f \circ \Phi$.

a) Show that in general Φ^* is neither injective nor surjective.

b) Prove that if Φ is a closed imbedding then Φ^* is surjective.

2. Consider the vector field $\mathbf{v} = \dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$.

a) Find the integral curve of $\mathbf{v}$ through $(a, b) \in \mathbb{R}^2$.

b) Find a smooth map $\mathbb{R}^2 \to \mathbb{R}$ such that the fibers are given by the integral curves of $\mathbf{v}$.

c) Find a 1-form $\mathbf{w}$ which annihilates $\mathbf{v}$. Is $\mathbf{w}$ exact ?

3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth manifold structure. For vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$, let $\mathbf{a} \times \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle$ respectively denote the vector product and the inner product.

a) Let $\mathbf{n}$ be the outward normal vector on S^2. Given $\sigma \in \bigwedge^1(S^2)$ defined by

$$\sigma(X) = \langle [1, 1, 1], X \times \mathbf{n} \rangle$$

prove that $\sigma = i^*(\Sigma)$ where $i : S^2 \to \mathbb{R}^3$ is the identity imbedding and

$$\Sigma = (y - z)dx + (z - x)dy + (x - y)dz.$$

b) Find $\Omega \in \bigwedge^2(\mathbb{R}^3)$ such that the volume element $\mathbf{w} \in \bigwedge^2(S^2)$ can be written in the form $\mathbf{w} = i^*(\Omega)$.

c) Does there exist $\theta \in \bigwedge^1(\mathbb{R}^3)$ such that $\mathbf{w} = i^*(d\theta)$? Explain.

4. True or false ? Explain (give a counter example if appropriate).

a) There exists no compact smooth 2-manifold M which admits an immersion $M \to \mathbb{R}^2$.

b) Let M be the compact surface and Γ be the oriented curve given in the figure. If $\mathbf{w}$ is a 1-form such that $\int_\Gamma \mathbf{w} \neq 0$, then $\mathbf{w}$ is not a closed form.

c) Let M, N be smooth manifolds with $\dim(N) > \dim(M)$ and let $\Phi : N \to M$ be a non-constant smooth map. If for some $y \in M$ the set $\Phi^{-1}(y)$ is a smooth submanifold of N, then y is a regular value of Φ.

DIFFERENTIABLE MANIFOLDS, SEPTEMBER 2010
TMS EXAM

SEPTEMBER 24, 2010

8 **1.a)** Let $\omega = (xy)\ dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r, s, t) = (r - ts, r^2 s + t)$. Calculate $f^*(\omega)$, the pullback of ω by f.

8 **1.b)** Consider the vector field on the plane

$$X = 2x\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}.$$

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

9 **1.c)** Recall that $H^2_{DR}(S^2) = \mathbb{R}$, which is spanned by the volume form $\omega = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$. Using the fact that $H^1_{DR}(S^2) = 0$, show that ω cannot be written as a product of two one-forms $\omega = \alpha \wedge \beta$, which are both closed.

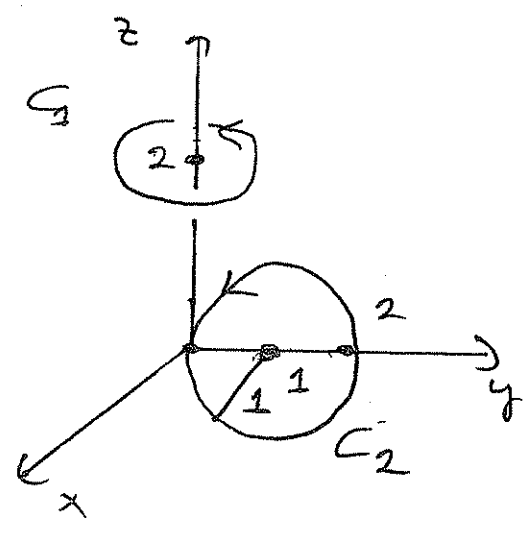
10 **2.a)** Let ω be the 1-form on $\mathbb{R}^3 - \{(x, y, z) \mid x^2 + y^2 - 1 = 0,\ z = 0\}$ given by

$$\omega = \frac{1}{2\pi}\frac{z\ d(x^2 + y^2 - 1) - (x^2 + y^2 - 1)\ dz}{((x^2 + y^2 - 1)^2 + z^2)^{1/2}}.$$

Show that ω is closed.

15 **2.b)** Calculate the integral of ω over the circles shown in the figure below.

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 2,\ x^2 + y^2 = 1\}\ ,$$
$$C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0,\ (y - 1)^2 + z^2 = 1\}.$$



/13 **3.a)** Prove that the subset $C = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x+1)\}$ is a smooth manifold by showing that $0 \in \mathbb{R}$ is a regular value for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y^2 - x(x-1)(x+1)$. What is its dimension? Describe its tangent space at any point $(a, b) \in C$.

/12 **3.b)** Similar to the **Part** (a) show that the unit sphere $S^2 \in \mathbb{R}^3$ is a smooth manifold of dimension two. Determine its tangent space at any point $(a, b, c) \in S^2$.

4) A one-form α on $\mathbb{R}^3$ is called a contact form if it satisfies

$$(\alpha \wedge d\alpha)(p)(e_1, e_2, e_3) > 0$$

at any point $p \in \mathbb{R}^3$, where e_i, $i = 1, 2, 3$, are the standard basis vectors in $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$.

8 a) Show that the one form $\alpha = x\ dy + dz$ is a contact form on $\mathbb{R}^3$.

9 b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a map given by

$$f(x, y, z) = (a_1 x + b, a_2 y, a_3 z),$$

where $a_1, a_2, a_3, b \in \mathbb{R}$, are some constants. Find necessary and sufficient conditions on these constants so that $f^*(\alpha) = \alpha$.

8 c) Show that a closed one-form ω on $\mathbb{R}^3$ cannot be a contact form.

M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

Geometry

Duration : 3 hr.

Each question is 25 pt.

1. **a)** Let $\omega = (x+y)\ dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r,s,t) = (r - t + s, e^r + t)$. Calculate $f^*(\omega)$, the pullback of ω by f.

   b) Consider the vector field on the plane

   $$X = 2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}.$$

   Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

   c) Calculate the bracket of the vector fields, $[X,Y]$, where $X = 2\dfrac{\partial}{\partial x} - xy\dfrac{\partial}{\partial y}$ and $Y = e^y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$.

2. **a)** Consider the real projective plane as the quotient space

   $$P : S^2 \to \mathbb{R}P^2 = S^2/\sim,\ (x,y,z) \mapsto [x:y:z],$$

   where $\sim$ is the equivalence relation on the unit two sphere S^2 defined by, $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ if and only if $(x_1,y_1,z_1) = -(x_2,y_2,z_2)$. Show that

   $$F : \mathbb{R}P^2 \to \mathbb{R}^5,\ [x:y:z] \mapsto (x^2, y^2, xy, yz, zx),$$

   is a smooth embedding.

   b) Let $\sigma : S^2 \to S^2$ be the antipodal map given by

   $$\sigma(x,y,x) = -(x,y,z).$$

Show that for the above map $P : S^2 \to \mathbb{R}P^2$ we have $P = P \circ \sigma$. Let $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$ a 2-form on S^2. Prove that $\omega \neq P^*(\nu)$, for any 2-form ν on the real projective plane.

3. **a)** Let $f : K \to \mathbb{R}^n$ and $g : L \to \mathbb{R}^n$ be embeddings of smooth manifolds, so that $\dim K + \dim L < n$. Consider the smooth mapping

$$\phi : K \times L \to \mathbb{R}^n,\ (p, q) \mapsto f(p) - g(q),\ (p, q) \in K \times L\ .$$

Show that a vector $v \in \mathbb{R}^n$ is a regular value for ϕ is and only if the images of the maps $f : K \to \mathbb{R}^n$ and

$$g + v : L \to \mathbb{R}^n,\ q \mapsto g(q) + v$$

are disjoint.

b) Let $f : S^1 \to \mathbb{R}^3$ and $g : S^1 \to \mathbb{R}^3$ be embeddings of the circle into $\mathbb{R}^3$. Using Part (a) conclude that for any $\epsilon > 0$ there is a vector $v \in \mathbb{R}^3$ with $\|v\| < \epsilon$, so that the embedded circles $f(S^1)$ and

$$g(S^1) + v = \{g(q) + v \mid q \in S^1\}$$

are disjoint.

4. A two-form ω on an oriented smooth four manifold, M^4, is called symplectic if it is both closed, $d\omega = 0$, and satisfies

$$(\omega \wedge \omega)(p)(e_1, e_2, e_3, e_4) > 0,$$

at any point $p \in M$, where e_i, $i = 1, 2, 3, 4$, are any set ordered basis (giving the chosen orientation of the manifold) vectors in T_pM^4.

a) Show that the two form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is a symplectic form on $\mathbb{R}^4$.

b) Show that the above form satisfies $\omega = d\alpha$, for the 1-form

$$\alpha = x_1\ dx_2 + x_3\ dx_4\ .$$

c) Show that a symplectic form ν on a compact oriented four dimensional manifold, M^4, cannot be an exact form (Hint: Use Stokes theorem).

METU MATHEMATICS DEPARTMENT
DIFFERENTIABLE MANIFOLDS
SEPTEMBER 2012 - TMS EXAM

SEPTEMBER 17, 2012

1.) Let $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = (x^2 + y^2 + z^2 - r^2 + 1)^2 - 4(x^2 + y^2) \ ,$$

where $0 < r < 1$ is a constant.

a) Show that $M = f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^3$.

b) Determine the tangent space $T_{(r+1,0,0)}M$ as a subspace of $T_{(r+1,0,0)}\mathbb{R}^3$.

2.) Consider the vector field on $\mathbb{R}^3$ given by

$$Y = (z - y)\frac{\partial}{\partial x} + (x - z)\frac{\partial}{\partial y} + (y - x)\frac{\partial}{\partial z} \ .$$

a) Show that the restriction of Y to the unit sphere $S^2 \subseteq \mathbb{R}^3$ defines a vector field on the unit sphere.

b) Determine the zeros of the vector field on the sphere.

3.) Consider the quotient topological space

$$M = \mathbb{R}^3 \ / \ (x, y, z) \sim (x + 1, y - 1, -z) \ , (x, y, z) \in \mathbb{R}^3 \ .$$

a) Show that M is a smooth manifold of dimension three.

b) Prove that M is not orientable showing that any 3-form on M has at least one zero.

4.a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Show that the 1-form

$$\omega = \frac{f\ dg - g\ df}{f^2 + g^2} \in \Omega^1(\mathbb{R}^n - Z) \ ,$$

where $Z = \{p \in \mathbb{R}^n \mid f(p) = 0 = g(p)\}$ is the set of common zeros of the functions f and g.

b) Let $\gamma : [0, 1] \to \mathbb{R}^n - Z$ be a smooth path such that $f(\gamma(t)) = 1$ for all $t \in [0, 1]$, and $g(\gamma(0)) = -1$ and $g(\gamma(1)) = 1$. Calculate the integral

$$\int_{[0,1]} \gamma^*(\omega) \ .$$

Differentiable Manifolds

TMS EXAM

September 16, 2013

Duration: 3 hr.

1. Find the tangent space to the surface $S:\ x^4 - y + z = 1$ at the point $p = (1, -1, 1)$ as a subspace of $\mathbb{R}^3$in two different ways:

(a) Using a local coordinate system at p.

(b) Exhibiting S as the preimage of a regular value under a map $f : \mathbb{R}^3 \to \mathbb{R}$ and then using the derivative of f (i.e. the induced map f_*).

2. Let $F : P^2(\mathbb{R}) \to P^1(\mathbb{R})$ be the map which is given by $F([x, y, z]) = [xy + x^2; y^2 + z^2]$. (Notation: The class of $x = (x_1, \ldots, x_{n+1})$ in $P^n(\mathbb{R})$ is denoted by $[x] = [x_1, \ldots, x_{n+1}]$.)

(a) Show that F is well defined.

(b) Choose a chart (U, ϕ) around a point $p = [x_0, y_0, z_0]$ in $P^2(\mathbb{R})$ with $y_0 \neq 0$ and a chart (V, ψ) around $F(p)$ with $F(U) \subset V$. Write the local expression of F in these charts. Is F smooth at p? Why?

(c) Compute the rank of the map F.

3. Consider the form $\omega = ydx - xdy$ in $\mathbb{R}^3$.

(a) Find the local expression of the restriction of this form to the cylinder $M : x^2 + y^2 = 1$ (i.e. $i * (\omega)$ where $i : M \to \mathbb{R}^3$ is the inclusion map) with respect to any chart of your choice.

(b) Let η be the form you have found in part (a). Find the local expression of $d\eta$ with respect to the chart you have used in part(a).

4. Let N be the unit ball in $\mathbb{R}^3$ and let f, g, h be smooth real valued functions defined on $\mathbb{R}^3$. Using Stokes Theorem write te the integral of $\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$ (more precisely the integral of the restriction of this form) over the boundary of N as an integral over N. Also write it as a (iterated) Riemannian integral.

5. Prove the following

(a) If $F : N \to M$ is a one-to-one immersion and N is compact, then F is an imbedding.

(b) If $F : N \to M$ is an immersion then each $p \in N$ has a neighborhood U such that $F|U$ is an imbedding of U in M.

METU MATHEMATICS DEPARTMENT
PRELIMINARY EXAMINATION
GEOMETRY MATH 505

SEPTEMBER 17, 2014

1.) Let ω be the closed 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

a) Calculate the integral $\displaystyle\int_{S^1} \omega$, where S^1 is the unit circle in the plane.

b) Use Stokes' Theorem to show that the integral $\displaystyle\int_C \omega = 0$, where $C = \{(x,y) \mid (x-5)^2 + y^2 = 1\}$.

c) Is there a smooth map $\phi : S^1 \times [0,1] \to \mathbb{R}^2 - \{(0,0)\}$, where $\phi(S^1 \times \{0\}) = S^1$ and $\phi(S^1 \times \{1\}) = C$, so that ϕ is a diffeomorphism when restricted to each of the boundary components of the cylinder? Justify your answer!

2.) Consider the Möbius band as the following quotient manifold

$$MB = \mathbb{R} \times (-1,1) \,/ (x,y) \sim (x+1,-y) \,.$$

a) Let $P : \mathbb{R} \times (-1,1) \to MB$ be the quotient map and

$$\sigma : \mathbb{R} \times (-1,1) \to \mathbb{R} \times (-1,1)$$

be the map given by $\sigma(x,y) = (x+1,-y)$. Show that for any smooth function $f : \mathbb{R} \times (-1,1) \to \mathbb{R}$ satisfying $f = -f \circ \sigma$, there is some $(x_0,y_0) \in \mathbb{R} \times (-1,1)$ with $f(x_0,y_0) = 0$.

b) Use Part (a) to show that for any 2-form ω on the Möbius band there is some $(x_0,y_0) \in \mathbb{R} \times (-1,1)$ with $\omega(P(x_0,y_0)) = 0$. Conclude that MB is not orientable.

3.) Show that the subset $\mathbb{R}^3$ given by

$$T^2 = \{(x,y,z) \in \mathbb{R}^3 \mid [(x^2+y^2+z^2)+3]^2 = 16(x^2+y^2)\}$$

is a submanifold. Show that it is diffeomorphic to the to the submanifold

$$\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid x_1^2 + y_1^2 = 1 = x_2^2 + y_2^2\}$$

via the map $F(x, y, z) = (\sqrt{x^2+y^2}-2, z, \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}})$. Determine F^{-1}.

4.) Let $\omega = f(x,y)dx + g(x,y)dy$ be a one-form on $\mathbb{R}^2 - \{(0,0)\}$.

a) Let C_R be the circle with center at the origin and radius $R > 0$, whose parametrization is given by $x = R\cos\theta$, $y = R\sin\theta$, $0 \leq \theta \leq 2\pi$. Assume that $|f(x,y)| \leq \frac{1}{\sqrt[4]{x^2+y^2}}$ and $|g(x,y)| \leq \frac{1}{\sqrt[4]{x^2+y^2}}$, for all $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$. Show that $|\int_{C_R} \omega| \leq 4\pi\sqrt{R}$.

b) Assume that the one-form ω is also closed. Use Stokes' theorem to show that $\int_{C_R} \omega = \int_{C_1} \omega$, for all $R > 0$.

c) Show that $\int_{C_R} \omega = 0$, for all $R > 0$. Conclude that ω is an exact form.

GEOMETRY TMS EXAM
October 01, 2015

Duration: 3 hours.

(1) Let $f : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be the map defined by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Consider $\mathbb{R}P^2$ as $S^2/\sim$ where $p \sim -p$ for all $p \in S^2$.
a) Write down a chart for $\mathbb{R}P^2$.
b) Let $F : \mathbb{R}P^2 \mapsto \mathbb{R}^4$ induced by f. Find F_*.
c) Is F embedding? Why?

(2) a) Show that the set $SL(2, \mathbb{R})$ of 2×2 real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?
b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = I$ may be identified with the set of matrices of zero trace.

(3) Let M be an even dimensional manifold, $dimM = 2n$. A differential form $\omega \in \Omega^2(M)$ is said to be non-degenerate if

$$\wedge^n \omega := \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$$

is a volume form. Show that on a compact orientable manifold M without boundary a non-degenerate 2-form ω cannot be exact.

(4) Let $\omega = \dfrac{xdy - ydx}{2\pi} \in \Omega^1(\mathbb{R}^2)$ and $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^k$, $k \in \mathbb{Z}_+$. Calculate

$$\int_{S^1} f^*(w).$$

(5) On $\mathbb{R}^4$ with coordinates (x, y, z, w) consider the following vector fields; $X_1 = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ and $X_2 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$ and 2-form $\omega = xdx \wedge dy + zdz \wedge dw$. Compute the following:
a) $[X_1, X_2]$
b) $d\omega$
c) $\Phi^*(\omega)$ where $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ is the map $\Phi(t, u) = (t \cos t, u, t \sin t, u)$.

GEOMETRY TMS EXAM
February 17, 2016

Duration: 3 hours.

(1) Show that $N = \{[x:y:z:w] \in \mathbb{R}P^3 | x^3 + y^3 + z^3 + w^3 = 0\}$ is an embedded submanifold of $\mathbb{R}P^3$, real projective space of dimension 3, and compute its dimension.

(2) Let M be an orientable smooth manifold and fix an orientation for unit circle S^1. Given a smooth map $\gamma : S^1 \longrightarrow M$ and a differential 1-form $\alpha \in \Omega^1(M)$ define $\int_\gamma \alpha := \int_{S^1} \gamma^*(\alpha)$.

a) Show that if α is exact then for any $\gamma : S^1 \longrightarrow M$,

$$\int_\gamma \alpha = 0.$$

b) Show that if $d\alpha = 0$, and $H : [0,1] \times S^1 \longrightarrow M$ is a smooth map then,

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha,$$

where $\gamma_0(\theta) = H(0,\theta)$ and $\gamma_1(\theta) = H(1,\theta)$.

(3)) Let $O(n)$ denotes the orthogonal $n \times n$ real matrices and $M(n)$ denotes $n \times n$ real matrices.

a) Show that the tangent space of $O(n)$ at the identity matrix, $T_I O(n)$ is the space of all anti-symmetric matrices.

b) Show that for any $A \in O(n)$, $T_A O(n) = \{XA | X^T = -X\}$.

c) Show that if $X \in T_I O(n)$ then $e^X \in O(n)$ where $e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$.

d) Consider the smooth map $exp : M(n) \longrightarrow M(n)$, defined as $exp(X) = e^X$. Show that the differential $dexp(\mathbf{0})$ at zero matrix $\mathbf{0} \in M(n)$ is the identity linear transformation.

(4)) Let Z be the preimage of a regular value $y \in Y$ under the smooth map $F : X \longrightarrow Y$ between smooth manifolds X and Y. Prove that the kernel of the derivative $dF_x : T_x X \longrightarrow T_y Y$ at any point $x \in Z$ is precisely the tangent space to Z at x, $T_x Z$.

(5)) Define $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$. On $\mathbb{R}^2$ with coordinates (u,v) consider the following vector fields; $U_1 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ and $U_2 = u\frac{\partial}{\partial u}$ and on $\mathbb{R}^3$ with coordinates (x,y,z) consider 2-form $\omega = ydx \wedge dz + xdy \wedge dz$ and 1-form $\eta = zdx + xdy + ydz$. Compute the following:

a) $F_*[U_1, U_2]$
b) $d\omega$
c) $F^*(d\eta)$
d) $F^*(\omega)(p)[V_1, V_2]$ where $V_1 = (1,2)$ and $V_2 = (0,1)$ are the vectors in $T_p\mathbb{R}^2$, for $p = (1,1) \in \mathbb{R}^2$
e) $\omega_{F(p)}(X_1, X_2)$ where $X_1 = F_*(V_1)$ and $X_2 = F_*(V_2)$.