1. Let $S^2$ be the unit circle in $\mathbb{R}^3$. Considering $S^2$ oriented by outer normal field
   
   a) exhibit a positively oriented basis of the tangent space for each point of $S^2$,  
   
   b) determine whether the reflection $F : S^2 \to S^2$ which is given by $F(x, y, z) = (x, -y, z)$ is orientation preserving or not.

2. Let $X, Y$ be smooth vector fields on a smooth manifold $M$. Then $XY$ defined by $(XY)(f) = X(Yf)$ makes sense as a smooth operator. We know that $[X, Y] = XY - YX$ is a smooth vector field.
   
   a) Show that $[fX, gY] = f g[X, Y] + f(Xg)Y - g(Yf)X$ for all smooth real valued functions $f$ and $g$ on $M$.
   
   b) Let $(U; x_1, \ldots, x_n)$ be a coordinate neighborhood on $M$ and let $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ be the associated coordinate frames. Show that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for each $i, j$ with $1 \leq i \leq n, 1 \leq j \leq n$.
   
   c) Assuming that dim $M = 2$, compute the components of $[X, Y]$ in terms of the components of $X$ and $Y$ with respect to a coordinate neighborhood.

3. Let $F : M \to N$ be a smooth map, $q \in N$ a regular value and $L = F^{-1}(q) \subset M$. Show that for any $p \in L$ the tangent space $T_p L$ is the kernel of the induced map $F_* : T_p M \to T_q N$.

4. Let $w$ be the 2-form on $\mathbb{R}^3 \setminus (0,0,0)$ given by $w = d\left(\frac{1}{x^2 + y^2 + z^2} dy\right)$.
   
   a) Find the local expression of the pull back of $w$ on $M$ with respect to the local parametrization
\[ \begin{align*}
  x &= 2 \cos u \ (1 + \cos v) - 2 \\
  y &= 2 \sin u \ (1 + \cos v) \\
  z &= \sin v \\
\end{align*} \]

\[ u, v \in (0, 2\pi). \]

b) Find \( \int_M w. \)
1. Consider the set \( M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1 , \ z^2 + w^2 = 1 \} \subseteq \mathbb{R}^4 \).

(a) Prove that \( M \) is an (imbedded) submanifold of \( \mathbb{R}^4 \).

(b) Describe the tangent vectors of \( M \) at an arbitrary point \((a, b, c, d) \in M\).

(c) Write down a nowhere vanishing vector field on \( M \).

(d) Let \( \omega = (y dx - x dy) \wedge (wdz - zdw) \in \Omega(\mathbb{R}^4) \). Show that \( \int_M i_*(\omega) > 0 \) where \( i : M \to \mathbb{R}^4 \) is the inclusion map (Hint: Write a local parametrization for \( M \)).

(e) A consequence of Poincaré Lemma is that every closed form on \( \mathbb{R}^n \) for any \( n \) is also exact. Prove that there exists no 4-form \( \theta \in \Omega(\mathbb{R}^4) \) with \( d\theta = 0 \) such that \( \int_M i^*(\theta) \neq 0 \).

2. Consider the the \((k - 1)\) dimensional sphere \( S^{k-1} \) as a submanifold of \( S^k \) via the usual embedding \((x_1, x_2, \ldots, x_k) \to (x_1, x_2, \ldots, x_k, 0)\). Show that the orthogonal complement to \( T_p(S^{k-1}) \) in \( T_p(S^k) \) is spanned by the vector \((0, 0, \ldots, 1)\).

3. Let \( \omega \) be a compactly supported 2-form
\[
w = f_1 \, dx_2 \wedge dx_3 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_1 \wedge dx_2
\]
on \( \mathbb{R}^3 \). Let \( S \) be the graph of a function \( G : \mathbb{R}^2 \to \mathbb{R} \). Compute the integral \( \int_S \omega \), and show that it is equal to \( \int_{\mathbb{R}^2} (\vec{F} \cdot \vec{n}) \|\vec{n}\| \, dx_1 \wedge dx_2 \) where \( \vec{F} = (f_1, f_2, f_3), \vec{n} = \frac{\vec{n}}{\|\vec{n}\|} \) with \( \vec{n} = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right) \).

4. Consider the sets
\[
M_1 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 + v^2 = w^2\} \subseteq \mathbb{R}P^2.
\]
\[
M_2 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 - v^2 = w^2\} \subseteq \mathbb{R}P^2.
\]
(a) Prove that $M_1$ is an (imbedded) submanifold of $\mathbb{R}P^2$ diffeomorphic to $S^1$ (Hint: Consider the image of $M_1$ under a suitable chart of $\mathbb{R}P^2$).

(b) Find a diffeomorphism $F: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ such that $F(M_1) = M_2$. 
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Geometry  

1.a. Let \( v_1 = (2, -3, -1) \), \( v_2 = (0, 4, 8) \) and \( v_3 = (2, 0, 0) \) be vectors in \( \mathbb{R}^3 \). Calculate \((dx \wedge dz)(v_1, v_2)\) and \((dx \wedge dy \wedge dz)(v_1, v_2, v_3)\).  

1.b. Let \( \omega = (-2x + y) \, dx \wedge dy \), a 2-form on \( \mathbb{R}^2 \), and \( f: \mathbb{R}^3 \to \mathbb{R}^2 \) be given by \( f(r, s, t) = (r - t, r^2 s) \). Calculate \( f^*(\omega) \), the pullback of \( \omega \) by \( f \).  

1.c. Repeat Part (b) for the constant function \( f(r, s, t) = (2, -5) \), for any \((r, s, t) \in \mathbb{R}^3\).  

2.a. Let \( \omega \) be the 2-form on \( \mathbb{R}^3 - \{(0, 0, 0)\} \) given by  
\[
\omega = \frac{1}{4\pi} \frac{x \, dy \wedge dz - y \, dx \wedge dz + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.
\]  
Show that \( \omega \) is closed.  

2.b. Calculate the integral of \( \omega \) over the 2-torus shown in the figure below. What would your answer be if the origin were inside the 2-torus?  

3.a. Show that the smooth map \( \Phi : S^2 \to \mathbb{R}^5 \), given by \( \Phi(x, y, z) = (x^2, y^2, xy, xz, yz) \) is an immersion, where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \).  

3.b. Show that \( \Phi \) is a 2-to-1 map with \( \Phi(x, y, z) = \Phi(-x, -y, -z) \). Conclude that \( \Phi \) gives a closed embedding of the real projective plane \( \mathbb{R}P^2 = S^2 / \sim \), where the equivalence relation \( \sim \) on \( S^2 \) is defined by, for \( p, q \in S^2 \) we have \( p \sim q \) if and only if \( p = -q \).
4.a. Show that 1 is a regular value of the smooth map $F : \mathbb{R}^4 \to \mathbb{R}$ given by $F(a, b, c, d) = ad - bc$. Conclude that the set of $2 \times 2$-matrices of determinant one, $SL(2, \mathbb{R})$, is a submanifold of the manifold of all $2 \times 2$-matrices $M(2, \mathbb{R}) = \mathbb{R}^4$. What is the dimension of $SL(2, \mathbb{R})$?

4.b. Is $0 \in \mathbb{R}$ a regular value of the same $F$? Justify your answer.
1.a) Let \( \omega = (x + yz) \, dx \wedge dy + dx \wedge dz \), a 2-form on \( \mathbb{R}^3 \), and \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be given by \( f(s, t) = (t + s, 2s + e^t) \). Calculate \( f^*(\omega) \), the pullback of \( \omega \) by \( f \).

1.b) Consider the vector field on the space
\[
X = 2x \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y} + xz \frac{\partial}{\partial z}.
\]
Calculate \( X(g) \) for any smooth function \( g : \mathbb{R}^3 \to \mathbb{R} \).

2.a) Let \( \omega \) be the 1-form on \( \mathbb{R}^3 \setminus \{(x, y, z) \mid x = 0, y = 0\} \) given by
\[
\omega = \frac{1}{2\pi} \frac{x \, dy - y \, dx}{x^2 + y^2}.
\]
Show that \( \omega \) is closed.

2.b) Calculate the integral of \( \omega \) over the circles shown in the figure below.
\[
C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\},
\]
\[
C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 1, y^2 + z^2 = 1\}.
\]
3.a) Prove that the subset $C = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 + 1\}$ is a smooth manifold by showing that $0 \in \mathbb{R}$ is a regular value for the function $f : \mathbb{R}^3 \to \mathbb{R}, f(x, y, z) = z^2 - x^2 - y^2 - 1$. What is its dimension? Describe its tangent space at any point $(a, b, c) \in C$.

3.b) Calculate the differential the smooth map $\Phi : M(n) \to S(n)$, $\Phi(A) = A^t A$, at the identity matrix $I_n$, where $M(n)$ is the set of all $n \times n$ matrices over reals and $S(n)$ is the set of symmetric real matrices over reals. Is the identity matrix $I_n$ a regular value for $\Phi$? (Hint: Note that we may regard $M(n)$ as $\mathbb{R}^{n^2}$ and $S(n)$ as $\mathbb{R}^{n(n+1)/2}$.)

4) Consider the 2-form on $\mathbb{R}^4$ $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

   a) Calculate $\omega \wedge \omega$.

   b) Can we write $\omega = \nu \wedge \eta$ for some 1-forms $\nu$ and $\eta$ on $\mathbb{R}^4$?

   c) Show that $\omega$ is closed. Let $S \subseteq \mathbb{R}^4$ be an embedded compact connected and orientable surface without boundary. Calculate the integral $\int_S \omega$. 
Duration: 3 hr.

1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = x^3 + xy + y^3 + 1.
\]
For which of the points \( p = (0, 0), p = (1/3, 1/3), p = (-1/3, -1/3) \) is \( f^{-1}(f(p)) \) an imbedded submanifold in \( \mathbb{R}^2 \)?

2. Let \( M \) be the hyperboloid of two sheets given by \( y^2 - z^2 - x^2 = 1 \).

(a) Let \( p \in M \). Explain how we can identify \( T_pM \) by a subspace of \( \mathbb{R}^3 \) using a chart at \( p \).

(b) Describe \( T_p(M) \) as a subspace of \( \mathbb{R}^3 \) if \( p = (0, 2, \sqrt{3}) \).

(c) Determine whether the map which assigns to each point \( q = (x, y, z) \) the vector \( (y, x + z, y) \) is a smooth vector field on \( M \).

3. Let \( F : M \to N \) be a smooth function between the manifolds \( M \) and \( N \) and let \( a \) be a smooth function on \( M \).

(a) Show that \( F^*(da) = d(F^*(a)) \)

(b) Verify the formula \( F^*d = dF^* \) on the forms of type \( \phi_1 \wedge \phi_2 \) where \( \phi_1 \) and \( \phi_2 \) are 1-forms.

(c) Let \( g : \mathbb{R}^3 \to \mathbb{R}^2 \) be given by
\[
g(x, y, z) = (xy, x^2yz)
\]
Compute \( g^*(2xydx \wedge dy) \)

4. Let
\[
\alpha = \frac{1}{2\pi} \frac{xydy - ydx}{x^2 + y^2}
\]

(a) Prove that \( \alpha \) is a closed 1-form on \( \mathbb{R}^2 \setminus 0 \)

(b) Compute the integral of \( \alpha \) over the unit circle \( S^1 \)?

(c) How does this shows that \( \alpha \) is not exact?
1. Let $S^2$ be the unit sphere in $\mathbb{R}^3$. Consider it with the topology relative to $\mathbb{R}^3$. Let $i : S^2 \rightarrow \mathbb{R}^3$ be the inclusion map.

(a) Show that $i$ is an immersion.

(b) Is $i$ an embedding? Why?

2. Let $M, N$ be two differentiable manifolds and $f : M \rightarrow N$ be a smooth map. Define a new map $F : M \rightarrow M \times N$ by $F(p) = (p, f(p))$.

(a) Show that $F$ is smooth.

(b) Show that $F_*(v) = (v, f_*(v))$ where $F_*$ and $f_*$ are induced maps at a point $p$ of $M$ and $v$ is a tangent vector of $M$ at $p$.

(c) Show that the tangent space to graph($f$) at the point $(p, f(p))$ is the graph of $f_* : T_pM \rightarrow T_{f(p)}N$.

3. Consider the 1-form $w = (x^2 + 7y)dx + (-x + y\sin y^2)dy$ on $\mathbb{R}^2$.

(a) Is $w$ exact? Is it closed?

(b) Compute the integral of $w$ over each side of the triangle whose vertices are $(0, 0), ((1, 0), (0, 2))$ where the sides are oriented in such a way that the triangle is oriented counterclockwise.

4. Let $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the map $F(p) = -p$.

(a) What is the induced map $F_*$? Why?

(b) Show that antipodal map $A : S^n \rightarrow S^n$ which is the restriction of $F$ on the $n$-sphere is orientation preserving if and only if $n$ is odd.

(c) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.
1. We identify $\mathbb{R}^4$ with the set of $2 \times 2$ real matrices.

(5 pts.) (a) Show that the set $SL(2, \mathbb{R})$ of $2 \times 2$ real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?

(5 pts.) (b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, may be identified with the set of matrices of zero trace.

2. (3 pts.) (a) Show that the 1-form $\omega = \frac{xdy - ydx}{x^2 + y^2}$ defined on $\mathbb{R}^2 - \{(0,0)\}$ is closed.

(3 pts.) (b) Calculate the integral $\int_{S^1} \omega$, where $S^1$ is the unit circle in $\mathbb{R}^2$.

(4 pts.) (c) Let $\Sigma$ be the smooth surface shown below with boundary $C$. Prove that there is no smooth map $\phi : \Sigma \to S^1$ such that $\phi|_C : C \to S^1$, the restriction of $\phi$ to the boundary $C$, is a diffeomorphism.

3. Let $f : X \to Y$ is a smooth map between manifolds, $f^*$ is the induced map between the algebras of differential forms of $X$ and $Y$ and $d$ is the exterior derivative.
(5 pts.) (a) Prove that \( d \circ f^* = f^* \circ d \).

(5 pts.) (b) If \( X = \partial W \) for some compact smooth manifold \( W \), and \( \omega \) is a closed \( n \)-form on \( Y \) with \( n = \dim X \), then show that

\[
\int_X f^*(\omega) = 0.
\]

4. (10 pts.) A curve in a manifold \( X \) is a smooth map \( t \mapsto c(t) \) of an interval of \( \mathbb{R}^1 \) into \( X \). The velocity vector of the curve \( c \) at time \( t_0 \) - denoted simply by \( \frac{dc}{dt}(t_0) \) - is defined to be the vector \( dc_{t_0}(1) \in T_{x_0}X \), where \( x_0 = c(t_0) \) and \( dc_{t_0} : \mathbb{R}^1 \to T_{x_0}X \) is the differential of \( c \) at \( t_0 \). In case \( X = \mathbb{R}^k \) and \( c(t) = (c_1(t), \ldots, c_k(t)) \) in coordinates, check that

\[
\frac{dc}{dt}(t_0) = (c'_1(t_0), \ldots, c'_k(t_0)).
\]

Prove that any vector in \( T_xX \) is the velocity vector of some curve in \( X \), and conversely.
1. Consider $(0, 2)$-tensor field $T$ and a $(1, 1)$-tensor field $S$ on $\mathbb{R}^2$, with the components $T_{i,j} = S_{j}^i = i - j + 2$, $i, j = 1, 2$, where $\mathbb{R}^2$ is considered as a manifold with usual coordinates (i.e. with coordinates with respect to the standard basis $e_1, e_2$)

(a) Determine the components $T_{\alpha\beta} S^{\beta}_\alpha$ of $T$ and $S$ when the coordinates in $\mathbb{R}^2$ are considered with respect to the basis $f_1 = e_1 + e_2$ and $f_2 = 2e_1 + e_2$

(b) Determine the components of $\text{Alt} T$ and $\text{Sym} T$ with respect to the basis $e_1, e_2$.

2. For each point $p = [u, v, w]$ on $\mathbb{R}P^2$ define curves $\gamma_p$ and $\sigma_p$ by

\[
\begin{align*}
\gamma_p(t) &= [u, e^{-t}v, e^{-t}w] \\
\sigma_p(t) &= [u \cos t - v \sin t, u \sin t + v \cos t, w]
\end{align*}
\]

for $t \in \mathbb{R}$. Consider the vector fields $A, B \in \mathfrak{X}(\mathbb{R}P^2)$ which assigns the values $\gamma'_p(0)$ and $\sigma'_p(0)$ respectively to each point $p \in \mathbb{R}P^2$

(a) Introduce a chart of your own choice on $\mathbb{R}P^2$ and find local expressions for $A, B$ on this chart.

(b) Find local expressions for the Lie bracket $[A, B]$ on the same chart.

(c) For each point $p = [u, v, w]$ on $\mathbb{R}P^2$ find a curve $\theta_p : \mathbb{R} \to \mathbb{R}P^2$ such that $\theta_p(0) = p$ and $[A, B]$ takes the value $\theta'_p(0)$ at the point $p \in \mathbb{R}P^2$.

3. Consider the two dimensional sphere

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} \subseteq \mathbb{R}^3$$
with its usual smooth structure and the smooth maps $f, g : S^2 \to \mathbb{R}$ defined by

\[
\begin{align*}
f((u, v, w)) &= w \\
g((u, v, w)) &= u
\end{align*}
\]

(a) Evaluate the integral

\[
\int_M df \wedge dg
\]

where $M$ is the manifold with boundary defined by

\[
M = \{(u, v, w) \in S^2 \mid v \geq 0\}
\]

without employing Stokes’ theorem.

(b) Use Stokes’ theorem to evaluate the same integral.

4. Let $M$ be a compact manifold and let $f : M \to N$ be a submersion where $N$ is an arbitrary manifold with $\dim M = \dim N$. Define a function $\phi : N \to \mathbb{R} \cup \{\infty\}$ by

\[
\phi(y) = \text{number of points in } f^{-1}(y)
\]

(a) Prove that $\phi(y)$ is finite for each $y \in N$.

(b) Prove that $\phi : N \to \mathbb{R}$ is a locally constant function.
1. a) Show that a one-to-one immersion of a **compact** manifold is an imbedding.
   
b) Explain, in full details, why the map \( \phi : (-\pi, \pi) \to \mathbb{R}^2, \phi(s) = (\sin(2s), \sin(s)) \) shows that the conclusion in part (a) is false if \( X \) is not compact.

2. Let \( SL_n(\mathbb{R}) \) denote the \( n \times n \) real matrices with determinant 1.
   
a) Show that \( SL_n(\mathbb{R}) \) is a submanifold of the \( n \times n \) matrices \( M_n(\mathbb{R}) \).
   
b) Show that the tangent space to \( SL_n(\mathbb{R}) \) at the identity matrix \( I \) is
   
   \[ T_I SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \text{trace}(A) = 0 \} \]

3. a) What is meant by an orientation on a manifold ?
   
b) Show that \( S^n = \{ \overline{x} \in \mathbb{R}^{n+1} : |\overline{x}| = 1 \} \) is an oriented manifold, by defining an orientation on it.
   
c) Show that the antipodal map \( S^n \to S^n, \overline{x} \mapsto -\overline{x} \) is orientation preserving if and only if \( n \) is odd.
   
d) Using (c), or otherwise show that \( \mathbb{R}P^n \) is orientable if and only if \( n \) is odd.

4. a) Show that \( X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \} \) is a closed submanifold of \( \mathbb{R}^3 \).
   
b) Verify that the restriction \( \omega|_X \) of \( \omega = \frac{x dy - y dx}{x^2 + y^2} \) is a closed 1-form on \( X \).
   
c) Calculate \( \int_S \omega|_X \), where \( S \) is the circle \( \{(x, y, 3) : x^2 + y^2 = 1 \} \subset X \).

   Is \( \omega|_X \) an exact form ? Why ?
   
d) Consider the mapping \( \Psi : \mathbb{R}^2 \to X, \Psi((s, t)) = (\cos(s), \sin(s), t) \). Show that \( \Psi \) is a differentiable map and that the form \( \Psi^*(\omega|_X) \) is exact.
Differentiable Manifolds

1. Let $\Phi : M \to N$ be a submanifold where $\dim(M) > 1$ and let

$$\Phi^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

be the restriction map $f \mapsto f \circ \Phi$.

a) Show that in general $\Phi^*$ is neither injective nor surjective.

b) Prove that if $\Phi$ is a closed imbedding then $\Phi^*$ is surjective.

2. Consider the vector field $v = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

a) Find the integral curve of $v$ through $(a, b) \in \mathbb{R}^2$.

b) Find a smooth map $\mathbb{R}^2 \to \mathbb{R}$ such that the fibers are given by the integral curves of $v$.

c) Find a 1-form $w$ which annihilates $v$. Is $w$ exact?

3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth manifold structure. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, let $\mathbf{a} \times \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle$ respectively denote the vector product and the inner product.

a) Let $\mathbf{n}$ be the outward normal vector on $S^2$. Given $\sigma \in \bigwedge^1(S^2)$ defined by

$$\sigma(X) = \langle [1, 1, 1], X \times \mathbf{n} \rangle$$

prove that $\sigma = i^*(\Sigma)$ where $i : S^2 \to \mathbb{R}^3$ is the identity imbedding and

$$\Sigma = (y - z)dx + (z - x)dy + (x - y)dz.$$

b) Find $\Omega \in \bigwedge^2(\mathbb{R}^3)$ such that the volume element $w \in \bigwedge^2(S^2)$ can be written in the form $w = i^*(\Omega)$.

c) Does there exist $\theta \in \bigwedge^1(\mathbb{R}^3)$ such that $w = i^*(d\theta)$? Explain.
4. True or false? Explain (give a counter example if appropriate).
   
a) There exists no compact smooth 2-manifold $M$ which admits an immersion $M \to \mathbb{R}^2$.

   b) Let $M$ be the compact surface and $\Gamma$ be the oriented curve given in the figure. If $w$ is a 1-form such that $\int_\Gamma w \neq 0$, then $w$ is not a closed form.

   c) Let $M$, $N$ be smooth manifolds with $\dim(N) > \dim(M)$ and let $\Phi : N \to M$ be a non-constant smooth map. If for some $y \in M$ the set $\Phi^{-1}(y)$ is a smooth submanifold of $N$, then $y$ is a regular value of $\Phi$. 
1. a) Let $\omega = (xy) \, dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r, s, t) = (r - ts, r^2 s + t)$. Calculate $f^*(\omega)$, the pullback of $\omega$ by $f$.

b) Consider the vector field on the plane

$$X = 2x \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y}.$$ 

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

c) Recall that $H^3_{dR}(S^2) = \mathbb{R}$, which is spanned by the volume form $\omega = x \, dy \wedge dz - y \, dx \wedge dz + z \, dx \wedge dy$. Using the fact that $H^3_{dR}(S^2) = 0$, show that $\omega$ cannot be written as a product of two one-forms $\omega = \alpha \wedge \beta$, which are both closed.

2. a) Let $\omega$ be the 1-form on $\mathbb{R}^3 - \{(x, y, z) \mid x^2 + y^2 - 1 = 0, \, z = 0\}$ given by

$$\omega = \frac{1}{2\pi} \frac{z \, d(x^2 + y^2 - 1) - (x^2 + y^2 - 1) \, dz}{((x^2 + y^2 - 1)^2 + z^2)^{1/2}}.$$ 

Show that $\omega$ is closed.

b) Calculate the integral of $\omega$ over the circles shown in the figure below.

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 2, \, x^2 + y^2 = 1\},$$

$$C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, \, (y - 1)^2 + z^2 = 1\}.$$
3.a) Prove that the subset \( C = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x+1)\} \) is a smooth manifold by showing that \( 0 \in \mathbb{R} \) is a regular value for the function \( f: \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = y^2 - x(x-1)(x+1) \). What is its dimension? Describe its tangent space at any point \((a, b) \in C\).

3.b) Similar to the Part (a) show that the unit sphere \( S^2 \in \mathbb{R}^3 \) is a smooth manifold of dimension two. Determine its tangent space at any point \((a, b, c) \in S^2\).

4) A one-form \( \alpha \) on \( \mathbb{R}^3 \) is called a contact form if it satisfies
\[
(\alpha \wedge d\alpha)(p)(e_1, e_2, e_3) > 0
\]
at any point \( p \in \mathbb{R}^3 \), where \( e_i, i = 1, 2, 3 \), are the standard basis vectors in \( T_p \mathbb{R}^3 \cong \mathbb{R}^3 \).

a) Show that the one form \( \alpha = x \, dy + dz \) is a contact form on \( \mathbb{R}^3 \).

b) Let \( f: \mathbb{R}^3 \to \mathbb{R}^3 \) be a map given by
\[
f(x, y, z) = (a_1x + b, a_2y, a_3z),
\]
where \( a_1, a_2, a_3, b \in \mathbb{R} \), are some constants. Find necessary and sufficient conditions on these constants so that \( f^*(\alpha) = \alpha \).

c) Show that a closed one-form \( \omega \) on \( \mathbb{R}^3 \) cannot be a contact form.
1. a) Let $\omega = (x + y) \, dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r, s, t) = (r - t + s, e^r + t)$. Calculate $f^*(\omega)$, the pullback of $\omega$ by $f$.

b) Consider the vector field on the plane

$$X = 2 \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y}.$$ 

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

c) Calculate the bracket of the vector fields, $[X, Y]$, where $X = 2 \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y}$ and $Y = e^y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$.

2. a) Consider the real projective plane as the quotient space

$$P : S^2 \to \mathbb{R}P^2 = S^2 / \sim, \quad (x, y, z) \mapsto [x : y : z],$$

where $\sim$ is the equivalence relation on the unit two sphere $S^2$ defined by, $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = -(x_2, y_2, z_2)$. Show that

$$F : \mathbb{R}P^2 \to \mathbb{R}^5, \quad [x : y : z] \mapsto (x^2, y^2, xy, yz, zx),$$

is a smooth embedding.

b) Let $\sigma : S^2 \to S^2$ be the antipodal map given by

$$\sigma(x, y, z) = -(x, y, z).$$
Show that for the above map \( P : S^2 \to \mathbb{R}P^2 \) we have \( P = P \circ \sigma \). Let \( \omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \) a 2-form on \( S^2 \). Prove that \( \omega \neq P^*(\nu) \), for any 2-form \( \nu \) on the real projective plane.

3. a) Let \( f : K \to \mathbb{R}^n \) and \( g : L \to \mathbb{R}^n \) be embeddings of smooth manifolds, so that \( \dim K + \dim L < n \). Consider the smooth mapping
\[
\phi : K \times L \to \mathbb{R}^n, \quad (p, q) \mapsto f(p) - g(q), \quad (p, q) \in K \times L.
\]
Show that a vector \( v \in \mathbb{R}^n \) is a regular value for \( \phi \) if and only if the images of the maps \( f : K \to \mathbb{R}^n \) and
\[
g + v : L \to \mathbb{R}^n, \quad q \mapsto g(q) + v
\]
are disjoint.

b) Let \( f : S^1 \to \mathbb{R}^3 \) and \( g : S^1 \to \mathbb{R}^3 \) be embeddings of the circle into \( \mathbb{R}^3 \). Using Part (a) conclude that for any \( \epsilon > 0 \) there is a vector \( v \in \mathbb{R}^3 \) with \( \|v\| < \epsilon \), so that the embedded circles \( f(S^1) \) and
\[
g(S^1) + v = \{ g(q) + v \mid q \in S^1 \}
\]
are disjoint.

4. A two-form \( \omega \) on an oriented smooth four manifold, \( M^4 \), is called symplectic if it is both closed, \( d\omega = 0 \), and satisfies
\[
(\omega \wedge \omega)(p)(e_1, e_2, e_3, e_4) > 0,
\]
at any point \( p \in M \), where \( e_i, \quad i = 1, 2, 3, 4 \), are any set ordered basis (giving the chosen orientation of the manifold) vectors in \( T_pM^4 \).

a) Show that the two form \( \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \) is a symplectic form on \( \mathbb{R}^4 \).

b) Show that the above form satisfies \( \omega = d\alpha \), for the 1-form
\[
\alpha = x_1 \, dx_2 + x_3 \, dx_4.
\]

c) Show that a symplectic form \( \nu \) on a compact oriented four dimensional manifold, \( M^4 \), cannot be an exact form (Hint: Use Stokes theorem).
1.) Let \( f : \mathbb{R}^3 \to \mathbb{R} \) by
\[
f(x, y, z) = (x^2 + y^2 + z^2 - r^2 + 1)^2 - 4(x^2 + y^2),
\]
where \( 0 < r < 1 \) is a constant.

a) Show that \( M = f^{-1}(0) \) is a smooth submanifold of \( \mathbb{R}^3 \).

b) Determine the tangent space \( T_{(r+1,0,0)}M \) as a subspace of \( T_{(r+1,0,0)}\mathbb{R}^3 \).

2.) Consider the vector field on \( \mathbb{R}^3 \) given by
\[
Y = (z - y)\frac{\partial}{\partial x} + (x - z)\frac{\partial}{\partial y} + (y - x)\frac{\partial}{\partial z}.
\]

a) Show that the restriction of \( Y \) to the unit sphere \( S^2 \subseteq \mathbb{R}^3 \) defines a vector field on the unit sphere.

b) Determine the zeros of the vector field on the sphere.

3.) Consider the quotient topological space
\[
M = \mathbb{R}^3 / (x, y, z) \sim (x + 1, y - 1, -z), (x, y, z) \in \mathbb{R}^3.
\]

a) Show that \( M \) is a smooth manifold of dimension three.

b) Prove that \( M \) is not orientable showing that any 3-form on \( M \) has at least one zero.

4.a) Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be smooth functions. Show that the 1-form
\[
\omega = \frac{f \, dg - g \, df}{f^2 + g^2} \in \Omega^1(\mathbb{R}^n - Z),
\]
where \( Z = \{ p \in \mathbb{R}^n \mid f(p) = 0 = g(p) \} \) is the set of common zeros of the functions \( f \) and \( g \).

b) Let \( \gamma : [0, 1] \to \mathbb{R}^n - Z \) be a smooth path such that \( f(\gamma(t)) = 1 \) for all \( t \in [0, 1] \), and \( g(\gamma(0)) = -1 \) and \( g(\gamma(1)) = 1 \). Calculate the integral
\[
\int_{[0,1]} \gamma^*(\omega).
\]
1. Find the tangent space to the surface $S: x^4 - y + z = 1$ at the point $p = (1, -1, 1)$ as a subspace of $\mathbb{R}^3$ in two different ways:

(a) Using a local coordinate system at $p$.

(b) Exhibiting $S$ as the preimage of a regular value under a map $f: \mathbb{R}^3 \to \mathbb{R}$ and then using the derivative of $f$ (i.e., the induced map $f_*$).

2. Let $F: P^2(\mathbb{R}) \to P^1(\mathbb{R})$ be the map which is given by $F([x, y, z]) = [xy + x^2, y^2 + x^2]$. (Notation: The class of $x = (x_1, \ldots, x_{n+1})$ in $P^n(\mathbb{R})$ is denoted by $[x] = [x_1, \ldots, x_{n+1}]$.)

(a) Show that $F$ is well defined.

(b) Choose a chart $(U, \phi)$ around a point $p = [x_0, y_0, z_0]$ in $P^2(\mathbb{R})$ with $y_0 \neq 0$ and a chart $(V, \psi)$ around $F(p)$ with $F(U) \subset V$. Write the local expression of $F$ in these charts. Is $F$ smooth at $p$? Why?

(c) Compute the rank of the map $F$.

3. Consider the form $\omega = ydx - xdy$ in $\mathbb{R}^3$.

(a) Find the local expression of the restriction of this form to the cylinder $M: x^2 + y^2 = 1$ (i.e., $i_* (\omega)$ where $i: M \to \mathbb{R}^3$ is the inclusion map) with respect to any chart of your choice.

(b) Let $\eta$ be the form you have found in part (a). Find the local expression of $d\eta$ with respect to the chart you have used in part (a).

4. Let $N$ be the unit ball in $\mathbb{R}^3$ and let $f, g, h$ be smooth real valued functions defined on $\mathbb{R}^3$. Using Stokes Theorem write the integral of $\omega = fdy \wedge dz + gdx \wedge dz + hdz \wedge dy$ (more precisely the integral of the restriction of this form) over the boundary of $N$ as an integral over $N$. Also write it as a (iterated) Riemannian integral.

5. Prove the following

(a) If $F: N \to M$ is a one-to-one immersion and $N$ is compact, then $F$ is an imbedding.

(b) If $F: N \to M$ is an immersion then each $p \in N$ has a neighborhood $U$ such that $F|U$ is an imbedding of $U$ in $M$. 
1.) Let $\omega$ be the closed 1-form

$$\omega = \frac{x \, dy - y \, dx}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

a) Calculate the integral $\int_{S^1} \omega$, where $S^1$ is the unit circle in the plane.

b) Use Stokes' Theorem to show that the integral $\int_C \omega = 0$, where $C = \{(x, y) \mid (x - 5)^2 + y^2 = 1\}$.

c) Is there a smooth map $\phi : S^1 \times [0, 1] \to \mathbb{R}^2 - \{(0, 0)\}$, where $\phi(S^1 \times \{0\}) = S^1$ and $\phi(S^1 \times \{1\}) = C$, so that $\phi$ is a diffeomorphism when restricted to each of the boundary components of the cylinder? Justify your answer!

2.) Consider the Möbius band as the following quotient manifold

$$MB = \mathbb{R} \times (-1, 1) / (x, y) \sim (x + 1, -y).$$

a) Let $P : \mathbb{R} \times (-1, 1) \to MB$ be the quotient map and

$$\sigma : \mathbb{R} \times (-1, 1) \to \mathbb{R} \times (-1, 1)$$

be the map given by $\sigma(x, y) = (x + 1, -y)$. Show that for any smooth function $f : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ satisfying $f = -f \circ \sigma$, there is some $(x_0, y_0) \in \mathbb{R} \times (-1, 1)$ with $f(x_0, y_0) = 0$.

b) Use Part (a) to show that for any 2-form $\omega$ on the Möbius band there is some $(x_0, y_0) \in \mathbb{R} \times (-1, 1)$ with $\omega(P(x_0, y_0)) = 0$. Conclude that $MB$ is not orientable.

3.) Show that the subset $\mathbb{R}^3$ given by

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid [(x^2 + y^2 + z^2) + 3]^2 = 16(x^2 + y^2)\}$$


is a submanifold. Show that it is diffeomorphic to the to the submanifold
\[ \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid x_1^2 + y_1^2 = 1 = x_2^2 + y_2^2\} \]
via the map \( F(x, y, z) = (\sqrt{x^2 + y^2 - 2}, z, \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}) \). Determine \( F^{-1} \).

4.) Let \( \omega = f(x, y)dx + g(x, y)dy \) be a one-form on \( \mathbb{R}^2 - \{(0, 0)\} \).
   
   a) Let \( C_R \) be the circle with center at the origin and radius \( R > 0 \), whose parametrization is given by \( x = R \cos \theta, y = R \sin \theta, 0 \leq \theta \leq 2\pi \).
   
   Assume that \( |f(x, y)| \leq \frac{1}{\sqrt{x^2 + y^2}} \) and \( |g(x, y)| \leq \frac{1}{\sqrt{x^2 + y^2}} \), for all \( (x, y) \in \mathbb{R}^2 - \{(0, 0)\} \). Show that \( |\int_{C_R} \omega| \leq 4\pi \sqrt{R} \).

b) Assume that the one-form \( \omega \) is also closed. Use Stokes' theorem to show that \( \int_{C_R} \omega = \int_{C_1} \omega \), for all \( R > 0 \).

c) Show that \( \int_{C_R} \omega = 0 \), for all \( R > 0 \). Conclude that \( \omega \) is an exact form.
(1) Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the map defined by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Consider $\mathbb{R}P^2$ as $S^2/\sim$ where $p \sim -p$ for all $p \in S^2$.
   a) Write down a chart for $\mathbb{R}P^2$.
   b) Let $F : \mathbb{R}P^2 \rightarrow \mathbb{R}^4$ induced by $f$. Find $F_*$.
   c) Is $F$ embedding? Why?

(2) a) Show that the set $SL(2, \mathbb{R})$ of $2 \times 2$ real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?
   b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = I$ may be identified with the set of matrices of zero trace.

(3) Let $M$ be an even dimensional manifold, $\dim M = 2n$. A differential form $\omega \in \Omega^2(M)$ is said to be non-degenerate if
   \[ \wedge^n \omega := \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M) \]
   is a volume form. Show that on a compact orientable manifold $M$ without boundary a non-degenerate 2-form $\omega$ cannot be exact.

(4) Let $\omega = \frac{xdy - ydx}{2\pi} \in \Omega^1(\mathbb{R}^2)$ and $f : S^1 \rightarrow S^1$ defined by $f(z) = z^k$, $k \in \mathbb{Z}_+$. Calculate
   \[ \int_{S^1} f^*(\omega). \]

(5) On $\mathbb{R}^4$ with coordinates $(x, y, z, w)$ consider the following vector fields; $X_1 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ and $X_2 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$ and 2-form $\omega = xdx \wedge dy + zdz \wedge dw$. Compute the following:
   a) $[X_1, X_2]$
   b) $d\omega$
   c) $\Phi^*(\omega)$ where $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is the map $\Phi(t, u) = (t \cos t, u, t \sin t, u)$. 

**Duration:** 3 hours.

GEOMETRY TMS EXAM
October 01, 2015
GEOMETRY TMS EXAM  
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Duration: 3 hours.

(1) Show that $N = \{[x : y : z : w] \in \mathbb{RP}^3 \mid x^3 + y^3 + z^3 + w^3 = 0\}$ is an embedded submanifold of $\mathbb{RP}^3$, real projective space of dimension 3, and compute its dimension.

(2) Let $M$ be an orientable smooth manifold and fix an orientation for unit circle $S^1$. Given a smooth map $\gamma : S^1 \to M$ and a differential 1-form $\alpha \in \Omega^1(M)$ define $\int_{\gamma} \alpha := \int_{S^1} \gamma^* \alpha$.

a) Show that if $\alpha$ is exact then for any $\gamma : S^1 \to M$,

$$\int_{\gamma} \alpha = 0.$$  

b) Show that if $d\alpha = 0$, and $H : [0, 1] \times S^1 \to M$ is a smooth map then,

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha,$$

where $\gamma_0(\theta) = H(0, \theta)$ and $\gamma_1(\theta) = H(1, \theta)$.

(3) Let $O(n)$ denotes the orthogonal $n \times n$ real matrices and $M(n)$ denotes $n \times n$ real matrices.

a) Show that the tangent space of $O(n)$ at the identity matrix, $T_I O(n)$ is the space of all anti-symmetric matrices.

b) Show that for any $A \in O(n)$, $T_A O(n) = \{X \in O(n) \mid X^T = -X\}$.

c) Show that if $X \in T_I O(n)$ then $e^X \in O(n)$ where $e^X = I + X + \frac{1}{2} X^2 + \frac{1}{6} X^3 + \cdots$.

d) Consider the smooth map $\exp : M(n) \to M(n)$, defined as $\exp(X) = e^X$. Show that the differential $d\exp(0)$ at zero matrix $0 \in M(n)$ is the identity linear transformation.

(4) Let $Z$ be the preimage of a regular value $y \in Y$ under the smooth map $F : X \to Y$ between smooth manifolds $X$ and $Y$. Prove that the kernel of the derivative $dF_x : T_x X \to T_y Y$ at any point $x \in Z$ is precisely the tangent space to $Z$ at $x$, $T_x Z$.

(5) Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u, v) = (u, v, u^2 - v^2)$. On $\mathbb{R}^2$ with coordinates $(u, v)$ consider the following vector fields, $U_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}$ and $U_2 = u \frac{\partial}{\partial u}$ and on $\mathbb{R}^3$ with coordinates $(x, y, z)$ consider 2-form $\omega = ydz \land dx + xdy \land dz$. Compute the following:

a) $F_*(U_1, U_2)$

b) $d\omega$

c) $F^*(dx_1)$

d) $F^*(\omega)(V_1, V_2)$ where $V_1 = (1, 2)$ and $V_2 = (0, 1)$ are the vectors in $T_p \mathbb{R}^2$, for $p = (1, 1) \in \mathbb{R}^2$

e) $\omega_{F(p)}(X_1, X_2)$ where $X_1 = F_*(V_1)$ and $X_2 = F_*(V_2)$. 