5.2.2007

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Calculate the QR factorization of the matrix

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

   (a) using Given's rotation,

   (b) Householder's reflection.

(c) Let A be a symmetric matrix and let λ and x be an eigenvalue-eigenvector pair for$A$ with $||x||_2 = 1$. Let P be an orthogonal matrix for which $Px = e_1 = [1, 0, \cdots, 0]^T$ and consider the similar matrix $B = PAP^T$.

i. Show that the first row and column of B are zero except for the diagonal element which equals λ.

ii. For the matrix

$$A = \begin{bmatrix} 2 & 10 & 2 \\ 10 & 5 & -8 \\ 2 & -8 & 11 \end{bmatrix}$$

with $\lambda = 9$ as an eigenvalue with associated eigenvector $x = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]^T$. Produce a Householder matrix P for which $Px = e_1$ and then produce $B = PAP^T$.

iii. By using the matrix B in part (b) find two other eigenvalues of A.

(d) Let $A \in \mathbf{R}^{n\times n}$ be a symmetric positive definite matrix, $b \in \mathbf{R}^n$, $x^* := A^{-1}b$ and assume that $x \in \mathbf{R}^n$ is some approximation of x^* with $r := b - Ax$ denoting the associated residual.

i. Show that

$$a_2(r) \leq \| x - x^* \|_2 \leq C_2(A) a_2(r),$$

where

$$a_2(r) := \frac{\| r \|_2^2}{\| A^T r \|_2}, \; C_2(A) := \sup_{y\neq 0} \frac{\| A^T y \|_2 \| A^{-1} y \|_2}{\| y \|_2^2}.$$

ii. Prove the estimate

$$C_2(A) \leq \frac{1}{2}(\kappa_2(A) + \kappa_2^{-1}(A)),$$

where $\kappa_2(A) := \| A^{-1} \|_2 \| A \|_2$.

[Hint: Use the Kantorovich inequality]

$$< By, y >< B^{-1}y, y > \leq \frac{(\lambda_{\min}(B) + \lambda_{\max}(B))^2}{4\lambda_{\min}(B)\lambda_{\max}(B)} \| y \|_2^4$$

for symmetric positive definite matrices $B \in \mathbf{R}^{n\times n}$.]

(e) Consider the linear system of equations $Ax = b$ where $b \in \mathbf{R}^n$ and A is an $n \times n$ matrix given by

$$A = \begin{bmatrix} 4 & 1 & \cdots & \cdots & \cdots \\ 1 & 4 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & 1 & 4 & 1 \\ \cdots & \cdots & \cdots & 1 & 4 \end{bmatrix},$$

i. Write down the Jacobi iteration in matrix form for this system.

ii. Find the rate of convergence of the Jacobi iteration.

iii. How many iterations are required to reduce the 2-norm error by a factor of 10^{-6}?

Hint: Eigenvalues of a $n \times n$ tridiagonal matrix

$$\begin{bmatrix} a & b & \cdots & \cdots & \cdots \\ c & a & b & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & c & a & b \\ \cdots & \cdots & \cdots & a & b \end{bmatrix},$$

are given as

$$\lambda_j = a + 2\sqrt{bc}\cos\frac{j\pi}{n+1}, \quad j = 1, \cdots, n$$

February, 2012

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Let A and B have order n matrices with A nonsingular. Consider solving the linear system

$$\begin{aligned} Ax_1 + Bx_2 &= b_1 \\ Bx_1 + Ax_2 &= b_2 \end{aligned}$$

with x_1, x_2, b_1 and $b_2 \in \mathbb{R}^n$.

   (a) Find necessary and sufficient conditions for convergence of the iteration method for $m \geq 0$

$$\begin{aligned} Ax_1^{(m+1)} &= b_1 - Bx_2^{(m)} \\ Ax_2^{(m+1)} &= b_2 - Bx_1^{(m)} \end{aligned}$$

   (b) Repeat part (a) for the iteration method for $m \geq 0$

$$\begin{aligned} Ax_1^{(m+1)} &= b_1 - Bx_2^{(m)} \\ Ax_2^{(m+1)} &= b_2 - Bx_1^{(m+1)} \end{aligned}$$

   (c) Compare the convergence rates of the two methods given in part (a) and part (b)

2. Consider the least square problems of minimizing

$$\rho^2(x) = \|b - Ax\|^2.$$

Here A is $m \times n$ matrix of rank n, $(m \geq n)$ and $\|\cdot\|$ is the Euclidean vector norm. Let

$$A = (Q_1 \quad Q_2) \begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR decomposition of A where Q_1, Q_2 and R are respectively, $m \times n, m \times (m-n)$, and $n \times n$.

   (a) Show that the solution of the least squares problem satisfies the QR equation $Rx = Q_1^T b$ and that the solution is unique. Further show that $\rho(x) = \|Q_2^T b\|$.

   (b) Use the QR equation to show that the least square solution satisfies the normal equations $(A^T A)x = A^T b$.

3. Given the vector $x = (0 \quad 0 \quad 0 \quad 3 \quad 4)^T$, by using Householder transformation, calculate a matrix Q for which Qx will have zeroes in its ***last two*** positions.

4. Let A be a nonsingular matrix whose leading principal submatrices are all nonsingular. Partition A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is , say $k \times k$.

(a) Is A_{11} singular or nonsingular? Explain.

(b) Show that there is exactly one matrix M such that

$$\begin{pmatrix} I_{k\times k} & 0 \\ -M & I_{(n-k)\times(n-k)} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}$$

(c) Give the explicit formulas for M and $\tilde{A}_{22}$

(d) Show that $\tilde{A}_{22}$ is nonsingular.

(e) Let $A_{11} = L_1U_1$ and $\tilde{A}_{22} = L_2U_2$ be the LU factorization of A_{11} and $\tilde{A}_{22}$, respectively. Find matrices L_{12} and U_{12} such that LU decomposition of A is

$$A = \begin{pmatrix} L_1 & 0 \\ L_{12} & L_2 \end{pmatrix} \begin{pmatrix} U_1 & U_{12} \\ 0 & U_2 \end{pmatrix}$$

February, 2014

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Prove that for invertible square real matrices A and B, the followings hold

   (a) $\left\|A^{-1} - B^{-1}\right\| \leq \left\|A^{-1}\right\| \|B - A\| \left\|B^{-1}\right\|$.

   (b) If $B = A + \delta A$ is the perturbed matrix and $\|\delta A\| \left\|B^{-1}\right\| = \delta < 1$ then
   $$\left\|A^{-1}\right\| \leq \frac{1}{1-\delta} \left\|B^{-1}\right\|$$
   and
   $$\left\|A^{-1} - B^{-1}\right\| \leq \frac{\delta}{1-\delta} \left\|B^{-1}\right\|.$$

   (c) If $x = A^{-1}b$ and $x + \delta x = (A + \delta A)^{-1} b$ then
   $$\|\delta x\| \leq \frac{\delta}{1-\delta} \|x + \delta x\|$$
   $$\|\delta x\| \leq \frac{\epsilon}{1-\epsilon} \|x\|$$
   where
   $$\delta = \|\delta A\| \left\|B^{-1}\right\| < 1, \qquad \epsilon = \|\delta A\| \left\|A^{-1}\right\| < 1.$$

   (Note: The norms for matrices and vectors are any compatible matrix and vector norms.)

2. (a) In iteratively solving the linear system $Ax = b$ ($\det A \neq 0$), we generate a sequence $x^{(k)}$ by the formula
   $$x^{(k+1)} = x^{(k)} + wP^{-1}r^{(k)}$$
   starting with some initial guess $x^{(0)}$. Here, P is a nonsingular matrix, $w > 0$ be a constant and $r^{(k)} = b - Ax^{(k)}$ is the residual vector. Show that the method converges if $w|\lambda|^2 < 2\alpha$ for any complex eigenvalue $\lambda = \alpha + i\beta$ of $P^{-1}A$.

   (b) Obtain a convergent iterative sequence using the method given in part (a) with a suitable choice of w for solving $Ax = b$ (carry out 2 iterations with $x^{(0)} = [0,\, 0]^T$ and P is the identity matrix) where
   $$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

3. Let $\{p_j(x)\}_{j=0}^k$ be the set of orthonormal polynomials (j-th degree) on an interval $[a, b]$ with the inner product $< h, g >= \int_a^b h(x)g(x)dx$ for continuous functions h and g on $[a, b]$ and the norm is

$$\|g\| =< g, g >^{1/2} .$$

If $p_k^*(x) = \sum_{j=0}^k < f, p_j > p_j(x)$ is the best least squares approximation to a continuous function $f(x)$ on $[a, b]$, then show that

(a) $\lim_{k\to\infty} \|f - p_k^*\| = 0,$

(b) $\|f - p_n^*\|^2 = \sum_{j=n+1}^{\infty} < f, p_j >^2.$

4. Let

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \quad -1 < \epsilon < 1.$$

(a) Find LU factorization of A.

(b) Find Cholesky factorization of A, if any exits.

(c) Give estimates based on Gerschgorin's theorem for the eigenvalues of A.

(d) Show that it is positive definite.

February, 2015

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the Jacobi and Gauss-Seidel iteration matrices and their eigenvalues λ_i and μ_i, respectively. Show that $\mu_{max} = \lambda_{max}^2$

2. Consider the real system of linear equations

$$Ax = b \qquad (1)$$

where A is non singular and satisfies

$$(v, Av) > 0$$

for all real v, where the Euclidean inner product is used here.

   (a) Show that $(v, Av) = (v, Mv)$ for all real v where $M = \frac{1}{2}(A + A^T)$ is symmetric part of A.

   (b) Prove that

$$\frac{(v, Av)}{(v, v)} \geq \lambda_{min}(M) > 0$$

   where $\lambda_{min}(M)$ is the minimum eigenvalue of M.

3. An overdetermined system $Ax = b$, $(m > n)$ is written as

$$\begin{bmatrix} R \\ 0 \end{bmatrix} x \approx \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $R \in \mathbb{R}^{n \times n}$, $b_1 \in \mathbb{R}^n$, $b_2 \in \mathbb{R}^{(m-n)}$, and $0 \in \mathbb{R}^{(m-n) \times n}$. Show that the least square solution x can be obtained from

$$Rx = b_1$$

and the residual vector $r = b - Ax$ satisfies

$$\|r\|_2^2 = \|b_2\|_2^2 \, .$$

4. (a) Describe the singular value decomposition (SVD) of the matrix $A \in \mathbb{C}^{m \times n}$. Include an explanation of the rank of A and how the SVD relates to the four fundamental subspaces

   R(A) Range of A, R(A*) Range of A*

   N(A) Nullspace of A, N(A*) Nullspace of A*

   (b) Perform SVD on the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$$

   (c) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse). Leave in factored form.

19.9.2007

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using any method you like, determine the reduced and full QR-factorization of A

---

2. Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}.$$

a can be positive or negative.

(a) Find the range of values of a will the Jacobi method always convergent by computing the exact eigenvalues of the iteration matrix ?

(b) Find the range of values of a will the Gauss-Seidel method always convergent by estimating the eigenvalues of the iteration matrix using Gerschgorin's circle theorem?

---

3. Assume that $A \in \mathbb{C}^{mxn}$ with $m > n$ has a full rank. Show how to choose ϵ so that the 2−norm condition number of

$$B = \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix} \qquad (I \; nxn \text{ identity matrix})$$

is equal to the square root of the 2−norm condition number of A.

(Hint: Consider B^*B and the singular value decomposition of A).

---

4. (a) Assume $A^T Ax = A^T b$ and $(A^T A + E)\hat{x} = A^T b$ with $2||E||2 \leq \sigma_n(A)^2$. Show that if $r = b - Ax$ and $\hat{r} = b - A\hat{x}$, then

$$\hat{r} - r = (I - A(A^T A + E)^{-1} A^T)Ax$$

and

$$||\hat{r} - r||_2 \leq 2\kappa_2(A)\frac{||E||_2}{||A||_2}||x||_2.$$

(b) Assume $A^T Ax = A^T b$ and $A^T A\hat{x} = A^T b + e$ where $||e||_2 \leq \epsilon ||A^T||_2 ||b||_2$ and A has full column rank. Show that

$$\frac{||x - \hat{x}||_2}{||x||_2} \leq \epsilon\kappa_2(A)^2 \frac{||A^2||_2 ||b||_2}{||A^T b||_2}.$$

FALL 2009

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. To produce QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -4 & 5 \end{bmatrix}.$$

   (a) use Householder transformation,

   (b) Gram-Schmidt orthogonalization.

2. For solving the linear system of equations $Ax = b$ you are given the iterative equations

$$x^{(k+1)} = Bx^{(k)} + c, \quad k = 0, 1, 2, \cdots$$

with an initial vector $x^{(0)}$ and a known vector c. The $n \times n$ iteration matrix B satisfies

$$\sum_{i=1}^{n} |b_{ij}| \leq M < 1, \qquad j = 1, 2, \cdots, n.$$

Show that the iteration converges to x for any starting vector $x^{(0)}$ (i.e. show that $\lim_{k \to \infty} e^{(k)} = \lim_{k \to \infty} |x^{(k)} - x| = 0$).

3. Let $Ax = b$, of order n, be uniquely solvable as $x = A^{-1}b$.

   (a) If $\hat{x}$ denotes the approximate solution of the system relative to small perturbations in the right hand side b, show that

   $$\frac{\| r \|}{\| A \|} \leq \| e \| \leq \| A^{-1} \| \| r \|$$

   where $e = x - \hat{x}$ (the error) and $r = b - A\hat{x}$ (the residual).
   $\| \cdot \|$ denotes any matrix or vector norm.

   (b) If C is the computed inverse of A, define the residual matrix by

   $$R = I - CA$$

   and show that

   $$\| e \| \leq \frac{\| Cr \|}{1 - \| R \|}$$

   where $e = x - \hat{x}$, $r = b - A\hat{x}$ and again $\hat{x}$ is the approximate solution to $Ax = b$.

---

4. Suppose that $Q = I + YTY^T$ is orthogonal where $Y \in R^{n \times j}$ and $T \in R^{j \times j}$ is upper triangular. Show that if $Q_+ = QP$ where $P = I - 2vv^T/v^Tv$ is a Householder matrix, then Q_+ can be expressed in the form $Q_+ = I + Y_+T_+Y_+^T$ where $Y_+ \in R^{n \times (j+1)}$ and $T_+ \in R^{(j+1) \times (j+1)}$ is upper triangular.

TMS

Spring 2010

NUMERICAL ANALYSIS I

1- Consider the problem of solving a nonsingular linear system $Ax = b$.

(a) If the entries of A are given with perturbations (error) δA such that

$$(A + \delta A)(x + \delta x) = b$$

where δx is the perturbation in the solution vector x due to the perturbations in the matrix A, **prove that**

$$||\delta x|| \leq \frac{\epsilon}{1-\epsilon}||x||$$

where $\epsilon = ||\delta A||\ ||A^{-1}|| < 1$ and $||\cdot||$ is any matrix or vector norm.

(b) If the right hand side vector b is given with perturbation δb as

$$A(x + \delta x) = b + \delta b$$

where δx now, is the perturbation in the solution x due to the perturbation in b, **prove that**

$$\frac{||\delta b||}{||A||} \leq ||\delta x|| \leq ||A^{-1}||\ ||\delta b||.$$

2- Let A and B are matrices with size $n \times n$ and A is nonsingular. Consider solving the linear system

$$Az_1 + Bz_2 = b_1$$

$$Bz_1 + Az_2 = b_2$$

with $z_1, z_2, b_1, b_2 \in R^n$ and b_1, b_2 are given. **Find the condition for convergence** of the iteration method

$$\begin{aligned} Az_1^{(m+1)} &= b_1 - Bz_2^{(m)} \\ & \qquad\qquad\qquad\qquad m \geq 0 \\ Az_2^{(m+1)} &= b_2 - Bz_1^{(m)} \end{aligned}$$

where m is the iteration number.

3- Consider the two-step Newton method for

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad x_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)}$$

where k is the iteration number ($k \geq 0$). Show that, the iteration converges **cubically** to a simple root ζ of $f(x) = 0$.

4- Let

$$A = \begin{bmatrix} 6 & 2 & 1 \\ 1 & -5 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

(a) Using Gershgorin's Theorem show that the eigenvalues of A satisfies the inequality $1 \leq |\lambda| \leq 9$.

(b) Using Gershgorin's Theorem prove that all eigenvalues of a diagonally dominant matrix are non-zero.

(c) Using **power method** with $x_0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ compute the absolutely largest eigenvalue and corresponding eigenvector approximately (perform 3 iterations).

September, 2011

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the following single shift QR algorithm applied to a matrix $A \in \mathbf{R}^{\mathbf{m}\times\mathbf{m}}$:

$$
\begin{aligned}
&T^{(0)} = A \\
&for \quad k = 1, 2, \ldots \\
&T^{(k-1)} - \sigma^{(k-1)} I = Q^{(k)} R^{(k)} \qquad (QR \quad \text{decomposition}) \\
&T^{(k)} = R^{(k)} Q^{(k)} + \sigma^{(k-1)} I \\
&end(k)
\end{aligned}
$$

where the shifts are given by $\sigma^{(k-1)} = (T^{(k-1)})_{m,m}$ for all $k \geq 1$. Assume that none of the shifts correspond to an eigenvalue of the matrix A and the diagonal coefficients of the matrices $R^{(k)}$ produced during the decomposition of $T^{(k-1)} - \sigma^{(k-1)} I$ are nonnegative. For $k \geq 1$ define the matrices

$$Q_k = Q^{(1)} Q^{(2)} \ldots Q^{(k)} \quad \text{and} \quad R_k = R^{(k)} R^{(k-1)} \ldots R^{(1)}.$$

(a) Show that the sequence of matrices $T^{(k)}$ generated by single shift QR algorithm is orthogonally similar to the initial matrix A.

(b) Show that

$$\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) = Q_k R_k \quad \text{for} \quad k \geq 1.$$

(c) Verify that the first column of Q_k is

$$Q_k e_1 = \frac{1}{(R_k)_{1,1}} \Big[\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) \Big] e_1$$

That is $Q_k e_1$ is essentially the vector obtained by using a shifted power iteration starting with e_1.

(d) Suppose A is symmetric. Verify that the last column of Q_k is

$$Q_k e_m = (R_k)_{m,m} \Big[\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) \Big]^{-1} e_m.$$

That is $Q_k e_m$ is essentially the vector obtained by using a shifted inverse power iteration starting with e_m.

2. Let A be an $m \times n$ matrix $(m \geq n)$.

   (a) Define the concept of a full QR factorization of A and the concept of a reduced QR factorization of A.

   (b) Describe the method of Householder triangulations for computing a reduced QR factorization of A.

   (c) Let A be an matrix $n \times n$ and let a_j be its j^{th}column. Use a QR factorization of A to show that

$$|detA| \leq \prod_{j=1}^{n} \|a_j\|_2.$$

3. Let $A \in \mathbf{R}^{\mathbf{n} \times \mathbf{n}}$ be symmetric positive definite. The iteration for solving the system $Ax = b$ is defined by

$$x^{k+1} = x^k - \alpha(Ax^k - b) \quad k \geq 0.$$

   Assume that the eigenvalues of A satisfy $0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$.

   (a) Show that the iteration converges if and only if $0 < \alpha < \dfrac{2}{\lambda_n}$.

   (b) Show that the optimal choice of α is $\dfrac{2}{\lambda_1 + \lambda_n}$.

   (*You might use 2-norm*).

4. Let $A \in \mathbf{C}^{\mathbf{m} \times \mathbf{n}}$ with $m > n$.

   (a) Show that $A^*A + \epsilon I$ is Hermitian and positive definite for every positive value of ϵ.

   (b) Assume A has full rank. Show how to choose ϵ so that the 2– norm condition number of

$$B = \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix},$$

   where I is a $n \times n$ identity matrix, is equal to the square root of the 2– norm condition number of A.

   (c) Use the reduced SVD of A to express the solution x to the problem

$$\min \left\| \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2,$$

   where b is a $m \times 1$ vector.

GOOD LUCK!!!

September, 2015

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the iterative method

$$x^{(k+1)} = Ax^{(k)} + b$$

where $A = \begin{bmatrix} 5 & 3/2 \\ 4 & 4 \end{bmatrix}$ and b is an arbitrary vector.

   (a) Does this iteration converge for arbitrary initial vectors, $x^{(0)}$.

   (b) Find α, if possible, so that the following iteration converges for arbitrary initial vectors, $x^{(0)}$:

$$\begin{aligned} y^{(0)} &= x^{(0)} \\ x^{(k+1)} &= Ay^{(k)} + b \\ y^{(k+1)} &= \alpha y^{(k)} + (1-\alpha)x^{(k+1)} \end{aligned}$$

   where A is the matrix given above.

2. Show that the singular values of the following matrices are the same as their eigenvalues

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 8 & 0 \\ 1 & 0 & 8 \end{bmatrix}.$$

(Hint:Compute the singular value decomposition of the matrices A and B).

3. The quantity A is going to be computed

$$A = \frac{x^3\sqrt{y}}{z^2}$$

with the values

$$x = 8.36, \quad y = 80.46, \quad z = 25.8$$

where the absolute errors are

$$e_x = 0.01, \quad e_y = 0.02, \quad e_z = 0.03, \text{ for } x, y, \text{ and } z, \text{ respectively.}$$

   (a) Find the upper bound for the relative error Rel_A of A.

   (b) Find the absolute error e_A in A.

4. Consider the $n \times n$, nonsingular matrix A. The Frobenious norm of A is given by

$$\|A\| = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{1/2}$$

(a) Construct the perturbation δA, with smallest Frobenious norm such that $A - \delta A$ is singular. (Hint: you might use of the primary decompositions of A)

(b) What is the Frobenious norm of this special δA?

(c) Prove that it is the smallest such perturbation?

September, 2016

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. (a) Suppose $A \in \mathbb{C}^{m\times m}$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{C}$ are eigenvalues of A. Prove that

$$tr(A) = \sum_{j=1}^{m} \lambda_j$$

   (b) Prove that, if A is Hermitian, then there is a unitary matrix U and a diagonal matrix D such that $U^*AU = D$

   (c) Prove that, if A is a real symmetric matrix, then there is an orthogonal matrix O and a diagonal matrix D such that $O^TAO = D$

2. We wish to solve $Ax = b$ iteratively where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   Show that for this A the Jacobi method and the Gauss-Seidel method both converge. Explain why for this A one of these methods is better than the other.

3. Suppose $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ denotes a matrix norm (Not necessarily induced by a vector norm) which also satisfies the compatibility property: for all $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$,

$$\|BC\| \leq \|B\|\|C\|.$$

   Let $\rho(A)$ denote the spectral radius of A.

   (a) Show that for $\rho(A) \leq \|A\|$ and then, that $\rho(A) \leq \|A^k\|^{1/k}$.

   (b) Show that for any given $0 < \epsilon << 1$,

$$\lim_{k\to\infty} \frac{\|A^k\|}{(\rho(A)+\epsilon)^k} = 0.$$

   Thus, conclude that $\lim_{k\to\infty} \|A^k\|^{1/k} \leq \rho(A)$.
   (Hint: You might use the fact that there exists an operator norm $\|A\|_{\epsilon,A}$ such that $\|A\|_{\epsilon,A} \leq \rho(A) + \epsilon/2$, for all $\epsilon > 0$)

   (c) From parts (a) and (b) what can you say about

$$\lim_{k\to\infty} \|A^k\|^{1/k}.$$

4. Let A be a real symmetric $n \times n$ matrix having the eigenvalues λ_1 with

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and the corresponding eigenvectors $x_1, \cdots, x_n$ with $\chi_i^T \chi_k = \delta_{ik}$. Starting with an initial vector y_0 for which $x_i^T y_0 \neq 0$, suppose one computes

$$y_{k+1} := \frac{1}{\| Ay_k \|} Ay_k \quad \text{for} \quad k = 0, 1, 2, \cdots$$

with an arbitrary vector norm $\| \cdot \|$, and concurrently the quantities

$$q_{ki} := \frac{(Ay_k)_i}{(y_k)_i}, \quad 1 \leq i \leq n, \quad \text{in case } \ (y_k)_i \neq 0,$$

and the Rayleigh quotient

$$r_k := \frac{y_k^T A y_k}{y_k^T y_k}$$

Prove the following:

(a) $q_{ki} = \lambda_1 [1 + O((\lambda_2/\lambda_1)^k)]$ for all i with $(x_1)_i \neq 0$.

(b) $r_k = \lambda_1 [1 + O((\lambda_2/\lambda_1)^{2k})]$.

February, 2017

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. (a) Show that the following matrix formula (where $q \in \mathbb{R}$) can be used to calculate $\mathbf{A^{-1}}$ when the process

$$x^{(n+1)} = x^{(n)} + q(\mathbf{A}x^{(n)} - \mathbf{I})$$

converges.

(b) When $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, give the values of q for which the process in (a) can be used. Which q yields the fastest convergence ?

(c) Let $\mathbf{A}$ be a symmetric and positive definite $n \times n$ matrix with smallest eigenvalue λ_1, and greatest eigenvalue λ_2. Find q to get as fast convergence as possible ?

[Hint: Fastest convergence is obtained when the spectral radius is minimized].

2. Given an $\mathbb{R}^{n\times m}$ matrix $\mathbf{A}$ with singular value decomposition $\mathbf{A} = \mathbf{UDV^T}$. Let $\mathbf{A}^\dagger$ be the pseudo-inverse matrix of $\mathbf{A}$.

(a) Verify that the singular value decomposition of $\mathbf{A}^\dagger$ is $\mathbf{A}^\dagger = \mathbf{VD^+U^T}$, where $\mathbf{D}^+$ is the transpose of $\mathbf{D}$ with every non-zero entry replaced by its reciprocal.

(b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$, find the SVD of $\mathbf{A}$.

(c) For the matrix $\mathbf{A}$ in part b), calculate $\mathbf{A}^\dagger$ through the SVD of $\mathbf{A}^\dagger$.

3. Let $\mathbf{x}$ be the solution of $A\mathbf{x} = \mathbf{b}$, where A is square and invertible. Carry out the perturbation analysis when *both* the matrix A and the vector $\mathbf{b}$ is perturbed. Let $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ such that $(A + \delta A)\tilde{\mathbf{x}} = \mathbf{b} + \delta\mathbf{b}$. Prove the following estimate:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}} \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right),$$

provided that δA is sufficiently small, in our case assume that $\|A^{-1}\| \cdot \|\delta A\| < 1$. The matrix norm is the induced norm obtained from the vector norm used and $\kappa(A) = \|A\| \cdot \|A^{-1}\|$.

4. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix where eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$. Suppose that $\beta \in \mathbb{R}$ and the vector $x \in \mathbb{C}^{\,n}$, $x \neq 0$ are such that $d = Ax - \beta x$. Then,

   (a) Show that

   $$\min_{1\leq\mu\leq n} |\beta - \lambda_\mu| \leq \frac{\|d\|_2}{\|x\|_2}.$$

   [Hint: Since A is Hermitian, the corresponding eigenvectors $x_1, x_2, \cdots, x_n$ form an orthonormal basis for $\mathbb{C}^{\,n}$].

   (b) Apply this result to the matrix

   $$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 7 \end{bmatrix}$$

   with $\beta = 12$ and $x = (0.9, 1, 1.1)^T$ where A has eigenvalues $\lambda_1 = 13$, $\lambda_2 = 4$ and $\lambda_3 = 2$.