5.2.2007

Graduate Preliminary Examination
Numerical Analysis II

Duration: 3 hours

1. Consider $f(x) = (x-a)^n$ for some positive integer n and some real number a

   (a) Find the sequence $\{x_i\}$ generated by the Newton and show that

   $$x_{i+1} - a = \left(1 - \frac{1}{n}\right)(x_i - a)$$

   (b) Find the order of convergence of the sequence $\{x_i\}$.

   (c) Is this order compatible with the order of Newton's method? Give an explanation.

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2. Calculate a third order interpolating polynomial through the points $(0,0), (1,-2)$ $(2,0)$ and $(3,12)$ using Newton's Forward Divided Difference method. Give the table of differences, and compute the error of approximation of the resulting polynomial for $x = 4$. Would you get a different result using Newton's backward divided differences?

3. Let $< h, g >= \int_a^b \omega(x)h(x)g(x)dx$ for $h(x)$ and $g(x)$ in $C[a,b]$ and $\omega(x)$ is a continuous positive weight function on (a,b). Let $||h|| =< h, h >^{1/2}$.

   (a) If $f(x) \in C[a,b]$, then the polynomial $p_n^*(x) \in P_n$ which satisfies $||f - p_n^*|| \leq ||f - p|| \quad \forall p(x) \in P_n$ is given by

$$p_n^*(x) = \sum_{j=0}^{n} < f, p_j > p_j(x)$$

   where $\{p_j(x)_{j=0}^n$ is the orthonormal set of polynomials generated by the Gram-Schmidt process with respect to the inner product given above (P_n is the set of $n-th$ degree polynomials).

   (b) Show that the remainder function $(f(x) - p_n^*(x))$ is orthogonal to every polynomial in P_n.

   (c) Show that

$$||f - p_n^*||^2 = ||f||^2 - \sum_{j=0}^{n} < f, p_j >^2 .$$

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4. Find an approximate formula for the evaluation of the integral

$$\int_0^1 f(x)x^{-1/2}dx$$

that is exact for all polynomial of degree one of the form

$$I(f) = c_1 f(0) + c_2 f(1).$$

Determine the Peano kernel and the error term.

February, 2012

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. The following integration formula is Gaussian quadrature type

$$\int_{-1}^{1} f(x)\,dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$

   (a) Derive this formula.

   (b) Determine a formula for the integration

$$\int_{a}^{b} f(t)\,dt$$

   (c) By using part (a) and (b), evaluate

$$\int_{0}^{\pi/2} t\,dt$$

2. Assume that f be a 3 times continuously differentiable function near a root α. Show that the iterative process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3}$$

   is a third order process for solving $f(x) = 0$.

3. Estimate the multiple integral

$$I = \int_{0}^{1} \int_{1}^{e^x} \left(x + \frac{1}{y}\right) dy dx$$

   numerically by using

   (a) Trapezoidal rule in both x and y directions.

   (b) *Composite Trapezoidal* rule in x direction and Trapezoidal rule in y direction.

4. By using Newton form of an interpolating polynomial show that

(a) If $p(x) \in \mathcal{P}_n$ (the set of all n-th degree polynomials) interpolates a function f at a set of $n+1$ distinct nodes $x_0, x_1, \ldots x_n$ and if t is a point different from the nodes, then

$$f(t) - p(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

(b) If $f \in C^n[a, b]$ and if $x_0, x_1 \ldots, x_n$ are distinct points in $[a, b]$ then there exists a point $\eta \in (a, b)$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\eta)}{n!}.$$

(c) If f is a polynomial of degree k, then for $n > k$,

$$f[x_0, x_1, \ldots, x_n] = 0.$$

February, 2014

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Solve the equation $f(x) = 2$ where $f(x)$ is defined by the following table

x	$f(x)$	Δ^1	Δ^2	Δ^3
0	0			
		1		
1	1		2	
		3		0
2	4		2	
		5		
3	9			

where $\Delta^1 = f(x_{i+1}) - f(x_i)$ is the forward difference of $f(x)$ at x_i.

2. (a) Show that the function

$$x_+^3 = \begin{cases} x^3 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is a cubic spline.

(b) Show that a cubic spline on the set $\{x_i\}_{i=0}^m$ has a unique representation

$$s(x) = p(x) + \sum_{i=1}^{m-1} c_i (x - x_i)_+^3$$

where $p(x)$ is a third degree polynomial.

3. Let $p(x) = \dfrac{1}{2}x^2 + a_1 x + a_0$.

(a) Use

$$\int_0^1 f(x)dx = \int_0^1 f(x)p''(x)dx$$

to derive a quadrature formula for

$$\int_0^1 f(x)dx$$

which involves only the values of f and f' at the end points.

(b) Show that the formula is exact if f is a polynomial of degree≤ 1.

(c) Find a_0 and a_1 so that the quadrature rule is exact for polynomials of degree≤ 3.

4. Consider two equivalent equations

$$x \ln x - 1 = 0, \qquad \ln x - \frac{1}{x} = 0$$

in the interval $[1, 2]$.

(a) Write the Newton iteration for both formulations.

(b) By considering Newton's method as fixed point iteration find the rate of convergence of both methods. Which method is faster? The root in the interval $[1, 2]$ is $x^* = 1.7632,\ \ln(x^*) = 0.5672$.

February, 2015

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. For finding the square root of 3, the nonlinear equation $f(x) = x^2 - 3 = 0$ is given. For each of the functions below determine whether the corresponding fixed-point iteration scheme

$$x_{k+1} = g_i(x_k), \quad i = 1, 2, 3, \quad k = 0, 1, 2, \cdots$$

is locally convergent to $\sqrt{3}$. Explain your reasoning in each case.

   (a) $g_1(x) = 3 + x - x^2$

   (b) $g_2(x) = 1 + x - \dfrac{x^2}{3}$

   (c) $g_3(x) = x + x^2 - 3$.

   Carry out 2 iterations with the convergent $g_i(x)$ to find $\sqrt{3}$ approximately correct to <u>two</u> decimal places. What is the order of convergence? Why?

2. Consider the numerical quadrature rule to approximate $\int_0^1 f(x)\,dx$ given by

$$\int_0^1 f(x)\,dx \approx af(0) + bf(x_1).$$

   (a) Find the maximum possible degree of precision you can attain by appropriate choices of a, b and x_1.

   (b) With such choices of a and b, approximate $\displaystyle\int_0^1 x^3\,dx$ and compare with the exact value.

3. Suppose $H(x)$ is a piecewise cubic polynomial interpolating a function $f(x)$ as follows:

$$H(\xi_i) = f(\xi_i), \quad H'(\xi_i) = f'(\xi_i), \quad i = 0, 1, \ldots, , N,$$

where ξ_i's form a partition of $[a, b]$ such that

$$\begin{aligned} a &= \xi_0 < \xi_1 < \cdots < \xi_N = b \\ h &= \xi_i - \xi_{i-1}, \quad i = 1, 2, \ldots, N \end{aligned}$$

Define $R(f; x)$ to be the error function given by

$$R(f; x) = f(x) - H(x)$$

and assume that $f(x)$ is in $C^4([a, b])$.

   (a) Show that

$$\frac{d^4}{dx^4} R(f; x) = \frac{d^4}{dx^4} f(x)$$

   (b) Show that for $x \in [\xi_i, \xi_{i+1}]$, there exists a $y \in (\xi_i, \xi_{i+1})$ such that

$$R(f; x) = \frac{(x - x_i)^2 (x - x_{i+1})^2}{4!} f^4(y)$$

   (c) Show that

$$\max_{a \le x \le b} |R(f, x)| \le Ch^4$$

4. Let

$$I_n(f) = \sum_{k=1}^{n} w_{n,k} f(x_n, k), \quad a \le x_{n,k} \le b \tag{1}$$

be a sequence of integration rules.

(a) Suppose

$$\lim_{n\to\infty} I_n(x^k) = \int_a^b x^k dx, \quad k = 0, 1, \ldots \tag{2}$$

and

$$\sum_{k=1}^{n} |w_{n,k}| \le M \quad n = 1, 2, \ldots \tag{3}$$

for some constant M. Show that

$$\lim_{n\to\infty} I_n(f) = \int_a^b f(x)dx$$

for all $f \in C[a, b]$. (Hint: use Weierstrass approximation theorem)

(b) Show that if all $w_{n,k} > 0$ then (2) implies (3).

21.9.2007

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Given f_i and f_i' at the points x_i, $i = 1, 2$.

   (a) Using Newton's divided difference formula, determine the cubic $P(x)$ such that

   $$P(x_i) = f_i, \qquad \text{and} \quad \frac{d}{dx}P(x_i) = f_i'.$$

   (b) Show that

   $$\int_{x_1}^{x_2} P(x)dx = (x_2 - x_1)\frac{f_1 + f_2}{2} + \frac{(x_2 - x_1)^2}{12}(f_1' - f_2').$$

   (c) What is the numerical use of formula such as that in part (b).

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2. Let $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x), \cdots$, be a sequence of orthogonal polynomials on an interval $[a, b]$ with respect to a positive weight funciton $w(x)$. Let $x_1, \cdots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and $a < x_1 < \cdots < x_n < b$.

   (a) Show that the Lagrange polynomials of degree $n-1$,

   $$L_j(x) = \prod_{k=1, k\neq j}^{n} \frac{(x - x_k)}{x_j - x_k}, \quad 1 \leq j \leq n$$

   for these points are orthogonal to each other, i.e.,

   $$\int_a^b w(x) L_j(x) L_k(x) dx = 0, \quad j \neq k.$$

   (b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \cdots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n-1$ which interpolates the function $f(x)$ at the zeros $x_1, \cdots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

   $$\| p_{n-1} \|^2 = \sum_{k=1}^{n} y_k^2 \| L_k \|^2$$

   in the weighted least squares norm. This norm is defined as follows: for any general function $g(x)$,

   $$\| g \|^2 = \int_a^b w(x) [g(x)]^2 dx.$$

---

3. Let $f(x) = x - e^{-x}$.

(a) Prove that $f(x) = 0$ has a root $r \in (0, 1)$.

(b) Let (x_n) be the Newton's sequence related to $f(x) = 0$ with $x_0 \geq 0$. Prove that

$$0 \leq x_{n+1} \leq 1 + \frac{x_0}{2^{n+1}}, \quad \text{for all} \quad n$$

(c) Take $x_0 = 10^{10}$. How many iterations are needed to have $x_n \leq \frac{3}{2}$? Set $e_n = x_n - r$. Why for such n we have $|e_n| \leq \frac{3}{2}$?

(d) Knowing that $e_{n+1} = \frac{e_n^2 f''(\theta_n)}{2f'(x_n)}$ with θ_n between x_n and r prove that

$$|e_{n+1}| \leq 2\left(\frac{e_0}{2}\right)^{2^{n+1}}.$$

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4. (a) Let us consider $f(x) = \alpha e^{-x}(1+x^2)^{1/2}$ in $\Omega = [0,1]$. For which values of α has $f(x)$ a unique fixed point in Ω.

(b) Apply Newton's method to the function $f(x) = 1/x - a$ to find $g(x)$ such that the iterates

$$x_{k+1} = g(x_k)$$

converge to $1/a$. Show that this iteration formula can be written in the interesting form

$$x_{k+1} f(x_{k+1}) = (x_k f(x_k))^2.$$

September, 2009

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. By using Newton form of an interpolating polynomial show that

   (a) If $p(x) \in P_n$ interpolates a function f at a set of $n+1$ distinct nodes $x_0, x_1, \cdots, x_n$ and if t is a point different from the nodes, then

   $$f(t) - p(t) = f[x_0, x_1, \cdots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

   (b) If $f \in C^n[a, b]$ and if $x_0, x_1, \cdots, x_n$ are distinct points in $[a, b]$ then there exists a point $\xi \in (a, b)$ such that

   $$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

   (c) If f is a polynomial of degree k, then for $n > k$

   $$f[x_0, x_1, \cdots, x_n] = 0.$$

   Note: P_n is the set of all n-th degree polynomials, $\prod$ denotes product notation $f[x_0, x_1, \cdots, x_n]$ is the n-th order divided difference of f, $f^{(n)}$ denotes n-th derivative of f.

---

2. Let $\phi_0(x), \phi_1(x), \phi_2(x), \cdots,$ be a sequence of orthogonal polynomials ($\phi_j(x)$ is a jth degree polynomial) on an interval $[a,b]$ with respect to a positive weight function $w(x)$. Let $x_1, \cdots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and

$$a < x_1 < \cdots, < x_n < b.$$

(a) Show that the Lagrange polynomials of degree $n-1$,

$$L_j(x) = \prod_{\substack{k= \\ k \neq j}} \frac{(x - x_k)}{(x_j - x_k)}, \quad 1 \leq < \leq n$$

for these points are orthogonal to each other, i.e.,

$$\int_a^b w(x) L_j(x) L_k(x) dx = 0, \quad j \neq k.$$

(b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \cdots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n-1$ which interpolates the function $f(x)$ at the zeros $x_1, \cdots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

$$||p_{n-1}||^2 = \sum_{k=1}^{n} y_k^2 \, ||L_k||^2$$

in the weighted least squares norm. This norm is defined as follows: for any general function $g(x)$.

$$||g||^2 = \int_a^b w(x) g(x)^2 dx.$$

3. Suppose s is a root of the equation $f(x) = 0$ with multiplicity 2 (double root).

(a) Show that Newton's method converges to this root linearly.

(b) Modify Newton's method such that the sequence $\{x_n\}$ obtained from Newton's iterations converges to s quadratically.

(c) By using Newton's method find $\sqrt[5]{32}$. Take starting value $x_0 = 1.8$ and carry out at most 3 iterations.

---

4. You are required to obtain numerical integration formulas for

$$\int_{-1}^{1} f(x)dx$$

(a) Using only $f(1), f'(-1)$ and $f''(0)$ find an approximation to $\int_{-1}^{1} f(x)dx$ which is exact for all quadratic polynomials. i.e. $\int_{-1}^{1} f(x)dx = Af(1) + Bf'(-1) + Cf''(0)$.

(b) Derive a 3- point Gaussian quadrature formula

$$\int_{-1}^{1} f(x)dx = A_0 f(x_0) + A_1 f(x_1) + A_3 f(x_3).$$

(c) Show that the formula obtained in part (b) is exact for all polynomials of degree 5.

September, 2015

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Let $x_0, x_1, \cdots x_n$ be distinct real numbers and $l_k(x)$ be the Lagrange's basis polynomials. **Show that:**

   (a) For any polynomial $p(x)$ of degree $(n+1)$,

   $$p(x) - \sum_{k=0}^{n} p(x_k) l_k(x) = \frac{1}{(n+1)!} p^{(n+1)}(x) \phi_n(x)$$

   where $\phi_n(x) = \prod_{k=0}^{n} (x - x_k)$.

   (b) If $x_0, \cdots, x_n$ are the roots of the Gauss-Legendre polynomial of degree $(n+1)$ in the interval $[-1, 1]$, then

   $$\int_{-1}^{1} l_i(x) l_j(x) dx = 0 \qquad \text{for} \quad i \neq j\,.$$

2. The function f has a continuous fourth derivative on $[-1, 1]$. Construct the **Hermite** interpolation polynomial of degree 3 for f using the interpolation points $x_0 = -1$ and $x_1 = 1$. Deduce that

   $$\int_{-1}^{1} f(x)\, dx = [f(-1) + f(1)] + \frac{1}{3}[f'(-1) - f'(1)] + E$$

   where

   $$|E| \leq \frac{2}{45} \max_{x \in [-1,1]} |f^{(4)}(x)|$$

3. Evaluate the following integral

   $$\int_{1}^{\infty} e^{-x} x^2 dx$$

   using proper Gaussian quadrature.

   Hint: You may take

   $$\sum_{i=1}^{n} A_i x_i^2 = 2 \quad , \quad \sum_{i=1}^{n} A_i x_i = 1 \quad , \quad \sum_{i=1}^{n} A_i = 1$$

   in your Gaussian quadrature where x_i and A_i are the points and weights of the integration, respectively.

4. Consider the fixed point iteration method

$$x_{n+1} = g(x_n) \tag{1}$$

(a) State the necessary conditions for existence and uniqueness of a fixed point $x = \alpha$ in (1), and deduce the criteria that determines the order of convergence.

(b) Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n} \tag{2}$$

Show that if α is a fixed point of $g(x)$, then it also a fixed point of $G(x)$.

(c) Consider the function $g(x) = x^2$, and deduce the convergence properties for both fixed point methods around the roots $x = 0$ and $x = 1$.

September, 2016

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. (a) Show that the function $f : (0, 1) \to (0, \infty)$ defined by $f(x) = -\ln x$ has a unique fixed point $s \in (0, 1)$.
   (b) Show, however, that fixed point iteration on $f(x)$ does NOT converge to s.
   (c) Reformulate the problem so that s is the unique fixed point of another function g for which the fixed point iteration converged to s for any $x_0 \in (0, 1)$.

2. (a) Write down the conditions that should be satisfied so that the following function is a **natural cubic spline** on the interval $[0, 2]$:

$$S(x) = \begin{cases} f_1(x) & : x \in [0, 1], \\ f_2(x) & : x \in [1, 2] \end{cases}$$

   (b) Determine the values of the coefficients a, b, c, d so that the following

$$S(x) = \begin{cases} x^2 + x^3, & : x \in [0, 1], \\ a + bx + cx^2 + dx^3 & : x \in [1, 2] \end{cases}$$

   is a **cubic spline** which has the property $S_1'''(x) = 12$.

3. Let $\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx$, where $w(x) \geq 0$ is a given weight function on $[a, b]$.
   (a) Prove that the sequence of polynomials defined below is orthogonal with respect to the inner product $\langle ., . \rangle$:

$$p_n(x) = (x - a_n) p_{n-1}(x) - b_n p_{n-2}(x), \quad n > 1,$$

   with

$$p_0(x) = 1,\ p_i(x) = x - a_i,$$
$$a_n = \langle x p_{n-1}, p_{n-1} \rangle / \langle p_{n-1}, p_{n-1} \rangle$$
$$b_n = \langle x p_{n-1}, p_{n-2} \rangle / \langle p_{n-2}, p_{n-2} \rangle.$$

   (b) Let $w(x) = 1 - x$ and $a = 0$, $b = 1$. Find the Gaussian quadrature for the integral $\int_0^1 (1 - x) f(x) g(x) dx$, which has algebraic degree of accuracy there. Use the general theory by constructing the corresponding orthogonal polynomials.

4. Let $f \in C^6[-1,1]$.

(a) Construct the Hermite interpolating polynomial $p(x)$ on the interval $[-1,1]$ such that

$$p(x_i) = f(x_i),\ \ p'(x_i) = f'(x_i) \ \text{ for } x_i = -1, 0, 1$$

(b) Give a formula for the interpolation error

$$E(f) = p(x) - f(x).$$

(c) Show that the quadrature formula

$$\int_{-1}^{1} f(t)dt \approx \frac{7}{15}f(-1) + \frac{16}{15}f(0) + \frac{7}{15}f(1) + \frac{1}{15}f'(-1) - \frac{1}{15}f'(1)$$

is exact for all polynomials of degree ≤ 5.

February, 2017

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Determine all values of a, b, c, d, e and f for which the following function $S(x)$ is a cubic spline

$$S(x) = \begin{cases} ax^2 + b(x-1)^3, & x \in (-\infty, 1], \\ cx^2 + d, & x \in [1, 2], \\ ex^2 + f(x-2)^3, & x \in [2, \infty). \end{cases}$$

2. (a) Derive a quadrature formula for

$$\int_{-1}^{1} x^2 f(x) \mathrm{d}x \approx \sum_{i=0}^{1} A_i f(x_i)$$

which is exact for polynomials of degree ≤ 3.

(b) Give an upper bound for the error made in the formula found in part (a).

(c) Evaluate $\int_0^2 (x-1)^2 e^x \mathrm{d}x$ by using the quadrature formula found in part (a).

3. (a) Use a suitable <u>*interpolating polynomial*</u> and its error term to derive the differentiation formula

$$f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{h^2}{6} f'''(\xi), \quad \xi \in (x_1 - h, x_1 + h)\,.$$

(b) Let $f(x) = e^{2x+1}$. Approximate $f'(1.4)$ by using the numerical differentiation formula obtained in part(a) and the values $f(1.3)$, $f(1.5)$, then approximate <u>the error</u>.

(c) The formula in part (a) can be written as

$$f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h} + K_1 h^2 + O(h^4)$$

where the constant $K_1 = -\frac{f'''(\xi)}{6}$. Use extrapolation to derive an $O(h^4)$ formula for $f'(x_1)$.

4. Given the polynomial $p(z) = z^4 + 2z^3 - 3z^2 + 2$.

(a) Locate the roots of $p(z)$ in the complex plane.

(b) Construct the synthetic division table for $p(z)$ for $z_0 = 2$; that is, write $p(z)$ as

$$p(z) = q_3(z)(z-2) + r_0,$$
$$q_3(z) = q_2(z)(z-2) + r_1$$
$$q_2(z) = q_1(z)(z-2) + r_2,$$
$$q_1(z) = q_0(z)(z-2) + r_3.$$

(c) Write $p(z)$ in Taylor series expansion around $z_0 = 2$ using part (b).

(d) If $x_0 = 2$ is an initial estimate to one of the real zeros of $p(z)$, carry out one iteration in Newton's method to approximate the root x_1 by using part (c).