Graduate Preliminary Examination
Ordinary Differential Equations
(3 hours)

February 16, 2011

1. Let $y_1 = y_1(x) = x^2$ and $y_2 = y_2(x) = x^5$.

   (a) Verify that y_1 and y_2 are linearly independent functions on $(0, \infty)$.

   (b) Find functions $h_1(x)$ and $h_2(x)$ such that $\{y_1, y_2\}$ is a fundamental set of solutions for the second order linear homogeneous equation

   $$y'' + h_1(x)y' + h_2(x)y = 0.$$

   (c) Solve the equation found in part (b) with the initial condition $y(1) = 1, y'(1) = -2$. Hint: Use the general solution.

2. Consider the following scalar differential equation

   $$x'(t) = x(\ln x)^2. \tag{1}$$

   (a) Determine the equilibrium, $x^* > 0$, of the equation (1).

   (b) Find the explicit solution $x(t, 0, x_0)$ and determine its domain.
   Consider $(a)\ \ 0 < x_0 < x^*, (b)\ \ x_0 = x^*, (c)\ \ x_0 > x^*$.

   (c) Can you investigate stability of the equilibrium , x^*, by applying the linearization technique? If not, why?

   (d) Can you investigate the stability by some another method? Show that the equilibrium is unstable.

3. Consider the system

   $$\frac{dz}{dt} = A(t)z + f(z) \tag{2}$$

   where $z \in R^n, A(t)$ is a continuous periodic, $n \times n$, matrix of period ω, $f(z)$ is continuous in some region about $z = 0$, and all Floquet multipliers of the system

   $$\frac{dy}{dt} = A(t)y \tag{$*$}$$

   are inside of the unit circle.

(a) The state transition matrix $\Phi(t,s) = \Psi(t)\Psi^{-1}(s)$ of (*) satisfies

$$||\Phi(t,s)|| < Ke^{-\alpha(t-s)}, t \geq s,$$

where K and α are some positive numbers, and $\Psi(t)$ is a fundamental matrix of (*).

(b) If the inequality $||f(z)|| \leq L||z||$ is valid, and $KL < \alpha$, then the trivial solution, $z \equiv 0$, of (2) is asymptotically stable.

Hint: Apply the Floquet theory and Gronwall inequality.

4. (a) State the Sturm-Picone comparison theorem.

(b) Show that every solution of

$$x'' + \left(1 - \frac{1}{t}\right)x = 0$$

has a sequence of zeros $\{t_n\}$ that is unbounded from above.

(c) Show also that $\lim_{n\to\infty} |t_n - t_{n-1}| = \pi$.

M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – February, 2014

Last Name :

Name :

Q.1 Consider the linear ODE: $x' = a(t)x + b(t)$, where $a(t)$ and $b(t)$ are continuous real functions on $t \geq 0$. Prove the following statements:

(a) The solution, satisfying $x(t_0) = x_0 \in \mathbb{R}$ for any $t_0 \geq 0$, is given by

$$x(t) = x_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(u) e^{\int_{u}^{t} a(s)ds} du$$

(b) If $a(t) \leq -m < 0$ and $b(t)$ is bounded on $t \geq 0$, then any solution is bounded on $t \geq 0$.

(c) If $a(t) \geq m > 0$ and $b(t)$ is bounded on $t \geq 0$, then there exists one and only one solution bounded on $t \geq 0$, which is given by

$$x(t) = -\int_{t}^{\infty} b(u) e^{-\int_{t}^{u} a(s)ds} du$$

Q.2 Let $h(t) \in \mathcal{C}([0,\infty], \mathbb{R}^+)$ and let $g(x) \in \mathcal{C}((0,\infty), \mathbb{R}^+)$. Suppose that

$$\lim_{B\to\infty} \int_{A}^{B} \frac{\mathrm{d}x}{g(x)} = +\infty, \qquad A > 0.$$

Then consider the IVP: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = h(t)g(x),\ x(\tau) = \xi$ with $\tau \geq 0$ and $\xi > 0$.

(a) Show that all solutions can be continued to the right over the entire interval $\tau \leq t < \infty$.

(b) If $\int_0^\infty h(t)\mathrm{d}t < \infty$, show that any solution of the IVP has a finite limit as $t \to \infty$.

(c) If $\lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} \frac{\mathrm{d}x}{g(x)} = +\infty$, show that all solutions can be continued to the left until $t = 0$.

Q.3 Consider the linear system with constant coefficients

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= a_{11}x + a_{12}y \\ \mathrm{d}y/\mathrm{d}t &= a_{21}x + a_{22}y \end{aligned}$$

where the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ are purely imaginary.

(a) Show that all solutions are closed trajectories (ellipses) surrounding the origin in the xy-plane. **Hint:** First observe that the eigenvalues of $\mathbf{A}$ are purely imaginary if and only if $\mathrm{tr}\mathbf{A} = 0$ and $\det\mathbf{A} > 0$. Then deduce that the system can be converted into a single equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$, which is exact.

(b) Show that the equilibrium solution is stable.

Q.4 If a nontrivial solution $\phi(t)$ of $y'' + (A + B\cos 2t)y = 0$ has $2n$ zeros in $(-\pi/2, \pi/2)$ and if $A, B > 0$, show that $A + B \geq (2n-1)^2$.

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 2004/2

1. Consider the following IVP

$$y' = y\cos(x^2 + y^2), \quad y(0) = 1.$$

where $y, x \in R^1$.
a) Applying the "existence-uniquenes" theorem, determine a *specific* interval on which a unique solution is sure to exist.
b) Determine the *largest* possible interval (α, β) on which the solution is defined.
c) Explain why the solution of the IVP is always positive.
d) Is the solution strictly increasing over its interval of definition? Why? Why not?

2. a) Let $y(t)$ be a solution of $y'' - e^{-t}y = 0$.
Show that $y(t)$ can not vanish twice.

b) Prove that every solution of $y'' + (1 + a(t))y = 0$ has infinitely many zeros, if

$$\lim_{t\to\infty} a(t) = 0.$$

3. Suppose that all solutions of $y'' + a(t)y = 0$ are bounded. Show that if $\int_0^\infty |b(t)|dt < \infty$, then all solutions of $y'' + (a(t) + b(t))y = 0$ are also bounded.

4. Using Lyapunov function show that the zero solution of the system

$$\begin{aligned} x_1' &= -2x_1x_2^2 - x_1^3, \\ x_2' &= -x_2 + x_1^2x_2 \end{aligned} \tag{1}$$

is uniform asymptotically stable.

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2005/2

1. Consider the IVP

$$x_1' = -\frac{1}{t}x_2 + \sin t,$$

$$x_2' = \frac{1}{t}x_1 + \cos t,$$

$$x_1(1) = x_2(1) = 0.$$

Show that the IVP has a unique solution defined on $(0, \infty)$.

2. Let $p(t)$ be continuous function defined on $[1, \infty)$ such that

$$\int_1^\infty |p(t) - c|dt < \infty,\ c > 0.$$

(a). Show that all solutions of

$$y'' + p(t)y = 0 \tag{1}$$

are bounded on $[1, \infty)$. (Hint: rewrite the equation in the form $y'' + cy = (c - p(t))y$).
(b). What can you say about the stability of the zero solution?
(c). Show that all solutions of (1) need not to be bounded if $c = 0$. (Hint: $p(t) = \frac{a}{t^2}$).

3. Let $A(t)$ be a continuous matrix for all $t \in R$. Let $P(t)$ be the matrix solution of

$$X' = A(t)X.$$

Show that $P(t)P^{-1}(s) = P(t - s)$ for all $t, s \in R$, if and only if $A(t)$ is a constant matrix.

4. Consider the following scalar equation

$$x' = c(t)x, \tag{2}$$

where function $c(t) : R \to R$ is defined in the following way:

$$c(t) = \begin{cases} t, & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ 1 - t, & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

and $c(t)$ is $1-$ periodic.
(a). Prove that (2) does not have a nontrivial 1-periodic solution.
(b). Does the equation have a nontrivial solution with another period?

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 12.09.2006

1. Consider the differential equation

$$y'' + q(x)y = 0, \tag{1}$$

where $q : [\alpha, \beta] \to R$ is a continuous function such that $0 < m \leq q(x) \leq M$. Let $\{x_1, x_2, \ldots, x_n\}$ be the zeros of a solution $y(x)$ such that $\alpha \leq x_1 < x_2, < \ldots < x_n \leq \beta$.

Show that:

(a) $\frac{\pi}{\sqrt{M}} \leq x_{i+1} - x_i \leq \frac{\pi}{\sqrt{m}}, i = 1, 2, \ldots, n-1$;

(b) $\frac{\sqrt{m}}{\pi}(\beta - \alpha) < n + 1$.

2. Applying the differentiable dependence of solutions on the initial value estimate the deviation of a solution $y(t) = y(x, 0, y_0)$ of the equation $y' = y + \sin y$ on $[0, 1]$ if the initial value is changed from 0 to y_0 and $|y_0| < 0.01$.

3. (a) Find all values of a parameter $a \in R$ such that the system

$$x' = 2y - 4x + 1, \quad y' = 2x - y + a$$

has solutions bounded on R.

(b) Define all these bounded solutions.

(c) Are these solutions stable?

4. For the initial value problem

$$y' = \lambda + \cos y, y(0) = 0,$$

find an upper estimate for $|y(x, \lambda_1) - y(x, \lambda_2)|$ and deduce that $y(x, \lambda)$ is continuous.

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2008/2

1. Consider the system

$$\begin{aligned} x' &= -3x - 2y + \sin(t), \\ y' &= 2x - 3y + \cos(t). \end{aligned} \tag{1}$$

   (a) Evaluate the transition matrix $X(t,s)$ of the associated homogeneous system. Show that $\limsup_{t\to\infty} \|X(t,s)\| = \infty$; 0

   (b) Find the general solution $x(t,t_0,x_0)$ of the system;

   (c) Show that all solutions are bounded on $[0,\infty)$ functions;

   (d) Show that there exists a unique solution bounded on R;

   (e) Prove that the bounded solution is $2\pi-$periodic function.

   (f) Prove that each solution of the system is uniformly asymptotically stable.

2. ~~Evaluate~~ Estimate $|x(t,0,x_0,x_0^1)|, |x'(t,0,x_0,x_0^1)|$, for $t \in [0,T], T < \infty$, if $x(t) = x(t,0,x_0,x_0^1), x(0) = x_0, x'(0) = x_0^1$, is a solution of equation $x'' + \sin x = 0$. Consider $x_0 = 0.01, x_0^1 = -0.02, T = 10$.

   Hint: Use differentiability of solutions in initial value.

3. Assume that $u(t) \geq 0, v(t) > 0$, are continuous on $[t_0 - T, t_0], t_0 \in R, T > 0$, functions. Prove that the inequality

$$u(t) \leq c + \int_t^{t_0} u(s)v(s)ds, t \leq t_0$$

   implies

$$u(t) \leq ce^{\int_t^{t_0} v(s)ds},$$

   where $c \geq 0$ is constant.

4. Consider the following Abel's equation

$$y' = \sin(t) - y^3. \tag{2}$$

   where $t, y \in R$. Prove that as t increasing, each solution of (2) is attracted into the strip $|y| < 1 + \epsilon$, where ϵ is a fixed positive number, in a uniformly bounded time interval.

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TMS

Fall 2009

Ordinary Differential Equations

1. Consider the differential equation

$$y' = yg(t, y),$$

where g and $\frac{\partial g}{\partial y}$ are defined and continuous for all $(t, y) \in R^2$. Show that

   (a) if $y = y(t)$, $t \in (a, b)$, is a solution satisfying $y(t_0) = y_0 > 0$, $t_0 \in (a, b)$, then $y(t) > 0$ for all $t \in (a, b)$

   (b) if $y = y(t)$, $t \in (a, b)$, is a solution satisfying $y(t_0) = y_1 < 0$, $t_0 \in (a, b)$, then $y(t) < 0$ for all $t \in (a, b)$

2. Consider the linear system

$$\dot{x} = A(t)x,$$

where A is a continuous for all $t \in J \subseteq R$.

Let $\Phi(t, t_0)$ be a matrix solution of the above system satisfying $\Phi(t_0, t_0) = I$.

   (a) Show that $x(t) = \Phi(t, t_0)x_0$ is the unique solution of the system satisfying $x(t_0) = x_0$.

   (b) Show that $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$, $t_1 \in J$.

   (c) Show that $\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t, s)A(s)$, $t, s \in J$.

   (d) Let $X(t) := \Phi(t, 0)$, $0 \in J$. Show that $\Phi(t, t_0) = X(t - t_0)$ if and only if A is a constant matrix.

3. Find the Folquet multipliers for the periodic system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\cos t + \sin t}{2 + \sin t - \cos t} \end{bmatrix} x$$

and deduce that there is a periodic s9olution? What is the periodic solution?

4. Show that the zero solution of

$$\dot{x} = -2x + \frac{\sin t}{t^2 + 1}x^3$$

is uniformly asymptotically stable.

Graduate Preliminary Examination
Ordinary Differential Equations
(3 hours)

September 20, 2010

1. Let $x(t, t_0, x_0)$ and $z(t, \bar{t}_0, \bar{x}_0)$ denote, respectively, solutions of

$$x' = f(t, x), \qquad x(t_0) = x_0$$

and

$$z' = f(t, z) + g(t, z), \qquad z(\bar{t}_0) = \bar{x}_0$$

where

(i) $f \in C(D)$, $|f(t, x)| \leq M$ for some $M > 0$, f is Lipschitz in x with Lipschitz contant L;

(ii) $g \in C(D)$, $|g(t, x)| \leq K$ for some $K > 0$.

   (a) Show that

$$|x(t) - z(t)| \leq \left(|x_0 - \bar{x}_0| + (M + K)|t_0 - \bar{t}_0| + \frac{M}{L} \right) e^{L|t - t_0|} - \frac{K}{L}.$$

   (b) Use part (a) with $g(t, x) \equiv 0$ to prove that the solution $x(t) = x(t, t_0, x_0)$ is continuous in (t_0, x_0) for t in a compact subset of real numbers.

2. Consider the systems

$$x' = Ax \tag{1}$$

$$y' = Ay + f(t, y) \tag{2}$$

where A is an $n \times n$ constant matrix and $f(t, y)$ is a continuous function defined on $R \times R^n$. Suppose that there exists a continuous function $\alpha(t)$ such that

$$||f(t, y)|| \leq \alpha(t)||y||, \qquad \int_a^\infty \alpha(t)\, dt < \infty, \quad (a \in R).$$

Show that if all solutions of (1) are bounded, then so are all solutions of (2). What would you say if A were not a constant matrix?

3. Consider the equation

$$\left(\frac{1}{1 + t} x'\right)' + (1 + \sin t)x = 0$$

   (a) Show that every solution of the equation has at least one zero in $[0, \pi]$.

   (b) Show that there is a solution having at least two zeros in $[0, \pi]$. Is it possible for any solution to have more than two zeros on $[0, \pi]$?.

4. Let

$$A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}.$$

   (a) Find the fundamental matrix e^{At} of of the linear system

$$x' = Ax.$$

   (b) Use part (a) to show that all solutions are periodic. What is the common period of the solutions?

   (c) Show that the zero solution is stable. Is it asymptotically stable?

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 2012/1

(1) Consider the differential equation $x'' + \omega^2 x = h(t)$, where $h(t)$ is continuous on (t_1, t_2), ω is a non-zero real constant. Show that the general solution is given by

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{1}{\omega}\int_{t_0}^{t} \sin\omega(t-s)h(s)ds,$$

where A and B are real constants and $t_0 \in (t_1, t_2)$ is a fixed real number. Use the preceding formula to find an integral equation that is equivalent to the nonlinear differential equation $x'' + \omega^2 x = f(t, x)$.

(2) Consider the linear differential equation

$$x' = (t^m A_0 + t^{m-1}A_1 + \ldots + A_m)x,$$

where $A_i, i = 0, 1, \ldots, m$, are constant n by n matrices, $x \in R^n$. Assume that the eigenvalues of A_0 have negative real parts. Prove that the solution $x = 0$ is asymptotically stable.

Hint: Introduce a new independent variable $s = (m+1)^{-1}t^{m+1}$.

(3) Consider the IVP

$$x_1' = (-1 + \sin t)x_2 + \frac{x_1}{1 + x_2^2} + 5t$$
$$x_2' = 2x_1 + (2 + \cos t)\frac{x_2}{1 + x_1^2} - t$$
$$x_1(0) = 1, \quad x_2(0) = 0.$$

(a) Show that the IVP has a unique solution $x = x(t)$ defined on an interval $(-c, c)$ for some $c > 0$.

(b) Show that

$$\| x' \|_1 \leq 5 \| x \|_1 + 6t, \quad t \geq 0.$$

Recall that $\| y \|_1 = |y_1| + |y_2|$.

(c) Use part (b) and the fact that $D^+ \| x \|_1 \leq \| x' \|_1$ to show that the solution is defined for all $t \geq 0$. Here D^+x is the upper Dini derivative of x.

(d) What can you say if $t \leq 0$?

(4) Let a be a continuous function satisfying $a(t + 2\pi) = a(t)$ for all $t \in \mathbb{R}$. Consider

$$x' = a(t)x.$$

Note that $x(t) = e^{\int_0^t a(s)ds}$ is a solution.

(a) Verify the e Floquet theorem.

(b) Calculate the Floquet exponent and the Floquet multiplier. Is there a periodic solution?

(c) Find the related constant coefficient equation.

(d) Answer (b) and (c) in the special case $a(t) = \sin t$.

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2013/2

1. Consider differential equations:

   (i) $x' = x^2$, with initial condition $x(0) = x_0 > 0$,

   (ii) $x' = x^2 + 1$, with initial condition $x(0) = x_0$.

   a) Verify that the theorem on existence and uniqueness applies.

   b) Solve for an explicit solution.

   c) What is the maximal interval of the solution?

2. Find a bounded on R solution, $x^0(t)$, of the equation $x' = -x + \sin t$. Prove that

   (a) $x^0(t)$ is a unique bounded solution of the equation;

   (b) the bounded solution is $2\pi-$ periodic;

   (c) the bounded solution is uniformly asymptotically stable.

3. Let $A(t)$ be a continuous matrix for all $t \in R$. Let $P(t)$ be the matrix solution of

$$X' = A(t)X.$$

   Show that $P(t)P^{-1}(s) = P(t-s)$ for all $t, s \in R$, if and only if $A(t)$ is a constant matrix.

4. Consider the following scalar equation

$$x' = \cos x. \tag{1}$$

   (a) Find all equilibriums of the equation.

   (b) Investigate stability of the solutions by linearization.

M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – September, 2014

Last Name :

Name :

Q.1 Consider the linear ODE: $x' = a(t)x + b(t)$, where $a(t)$ and $b(t)$ are continuous real functions on $t \geq 0$. Prove the following statements:

(a) The solution, satisfying $x(t_0) = x_0 \in \mathbb{R}$ for any $t_0 \geq 0$, is given by

$$x(t) = x_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(u) e^{\int_{u}^{t} a(s)ds} du$$

(b) If $a(t) \leq -m < 0$ and $b(t)$ is bounded on $t \geq 0$, then any solution is bounded on $t \geq 0$.

(c) If $a(t) \geq m > 0$ and $b(t)$ is bounded on $t \geq 0$, then there exists one and only one solution bounded on $t \geq 0$, which is given by

$$x(t) = -\int_{t}^{\infty} b(u) e^{-\int_{t}^{u} a(s)ds} du$$

(d) If $\lim a(t) = -A$ with $A > 0$ and $\lim b(t) = B$ as $t \to \infty$, then any solution of the linear ODE satisfies $\lim x(t) = B/A$ as $t \to \infty$.

Q.2 Let $\mathbf{A}(t)$ be an $n \times n$ continuous matrix for all $t \in \mathbb{R}$. Let $\Psi(t)$ be a matrix solution of $\mathbf{X}' = \mathbf{A}(t)\mathbf{X}$ with $\Psi(0) = \mathbf{I}_n$. Show that $\Psi(t)\Psi^{-1}(s) = \Psi(t-s)$ for all $t, s \in \mathbb{R}$, if and only if $\mathbf{A}(t)$ is a constant matrix.

Q.3 Consider the nonlinear system

$$\begin{aligned} dx/dt &= y \\ dy/dt &= -w^2 \sin x - \gamma y \end{aligned}$$

where γ and w are real constants.

(a) Find the critical points (equilibrium solutions), and deduce that the origin is an isolated critical point of the system.

(b) Using the **linear approximation**, examine the stability properties of the critical point at the origin.

(c) Explain why the trajectories of the linear system are good approximations to those of the nonlinear system, at least near the origin.

M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – September, 2015

Last Name :

Name :

Q.1 Use *Sturm Comparison Theory* to find the least possible number of zeros of a nontrivial solution of $y'' + t^2 y = 0$ on $(0, 5\pi)$. At most, how many zeros can have such a solution on $[0, 5\pi]$?

Q.2 Use *Green's formula* to find the differential operator adjoint to

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + a_0(t),$$

where a_0 and a_1 are real valued continuous functions on $t \in [a, b]$. Hence show that L is NOT formally self-adjoint. Then determine a function $\mu(t)$ appropriately to see that the operator $\mu(t)L$ is formally self-adjoint.

Q.3 Let $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ be two fundamental matrices for the linear homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t)$ is an $n \times n$ continuous real matrix on $t \in (a, b)$.
(a) Show that there is an invertible constant matrix $\mathbf{C}$ such that $\mathbf{\Phi}^{-1}(t)\mathbf{\Psi}(t) = \mathbf{C}$.
(b) If $\mathbf{W}(t)$ is a fundamental matrix for the adjoint system $\mathbf{y}' = -\mathbf{A}^T(t)\mathbf{y}$, show also that $\mathbf{W}^T(t)\mathbf{\Phi}(t) = \mathbf{C}$.

Q.4 Consider the nonlinear system

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= 2y \\ \mathrm{d}y/\mathrm{d}t &= -4\cos(x + \pi/2) + y \end{aligned}$$

(a) Find the critical points (equilibrium solutions), and deduce that the origin is an isolated critical point of the system.
(b) Using the **linear approximation**, examine the stability properties of the critical point at the origin.
(c) Explain why the trajectories of the linear system are good approximations to those of the nonlinear system, at least near the origin.

METU MATHEMATICS DEPARTMENT
ODE
FEBRUARY 2016 - TMS EXAM

1. Determine stable and unstable subspaces of solutions of the system

$x' = Ax,$

where

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

2. Consider the initial value problem,

$x' = x^{3/4}, x(t_0) = x_0.$

Prove that there are infinitely many solutions for any couple $(t_0, x_0), t_0 \in \mathbb{R}$, $x_0 \geq 0$.

3. Analyze stability of all equilibriums of the pendulum equation

$x'' + k \sin x = 0,$

where $k > 0$ is a constant.

4. Investigate for orbital stability the solution $\xi = \sin t$ of the scalar equation

$x'' + \mu x'(x^2 + x'^2 - 1) + x = 0,$

where μ is a scalar parameter.

TMS. Differential Equations (ODE)

1. Draw integral curves (the phase portrait) of the scalar equation

$$\frac{dy}{dx} = \frac{x-y}{|x-y|}. \tag{1}$$

2. Consider the Riccati equation

$$\frac{dy}{dx} = y^2 + f(x), \tag{2}$$

where $f(x)$ is an $\omega-$periodic function. Prove that

$$\int_0^{\omega} (y_1(x) + y_2(x))dx = 0,$$

where y_1, y_2 are two $\omega-$periodic solutions of the equation (2).

3. Analyze Lyapunov stability of the following initial value problem,

$$\tfrac{dx}{dt} = \tfrac{a}{t}x,\ x(1) = 0,$$

where a is a real parameter.

4. Solve the equation

$$x^2\frac{d^2y}{dx^2} - 2y = 0,$$

with boundary conditions $a)\ y(1) = 1,\ \lim_{x\to\infty} y'(x) = 0, b)\ \lim_{x\to 0} y(x) = 0,\ y'(1) = 1.$

PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2017/1

1. Supppose that $x_1(t)$ and $x_2(t)$ are the solutions of $x' + c(t)x = 0$ with $x_1(t_1) = a, x_2(t_2) = b$, where a, b are constants and t_1, t_2, t are members of an interval $I \subset R$, and $c(t)$ is a continuous function. Solve the equation and show that $x_1(t) - x_2(t) \to 0$ as $t_1 \to t_2$ and $a \to b$ for all $t \in I$.

2. Consider the IVP
$$x' = t^2 + x^2, x(0) = 0, 0 \leq t \leq a, |x| < b.$$
Show that

   (i) the solution exists on $0 \leq t \leq \min(a, \frac{b}{a^2+b^2})$;

   (ii) the maximum value of $\frac{b}{a^2+b^2}$ is $1/(2a)$ for a fixed a;

   (iii) $h = \min(a, 1/(2a))$ is largest when $a = 1/\sqrt{2}$;

   (iv) discuss the maximum interval of existence on the basis of (ii) and (iii).

3. Solve the BVP,
$$y'' + y = 0, y(0) = 0, y(a) = y_0. \tag{1}$$

4. Consider the following scalar equation
$$x' = a(t)x, \tag{2}$$
where $a(t) : R \to R$ is a continuous function. Prove that the zero solution, $x \equiv 0$, of the equation is uniformly stable if and only if
$$\int_{t_0}^{t} a(s)ds \leq M < \infty, t \geq t_0 > 0,$$
with M constant.