19 February 2004

Graduate Preliminary Examination
Partial Differential Equations
Duration: 3 hours

1. Find the characteristics of the equation $u_x^2 + u_y^2 = u^2$ and determine the integral surface passing through $x^2 + y^2 = 1$, $u = 1$.

2. Find the solution $u = f(x^2 - c^2t^2) = f(s)$, where $f(0) = 1$, of

$$u_{tt} - c^2 u_{xx} = \lambda^2 u, \quad \lambda: \text{ constant}$$

in the form of a power series.

3. a) Show that the problem

$$\frac{\partial}{\partial x}(e^x \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(e^y \frac{\partial u}{\partial y}) = 0 \qquad \text{in } D$$

$$u = x^2 \qquad \text{on } \partial D$$

cannot have more than one solution if $u \in C(\overline{D}) \cap C^2(D)$ where

$$D := \{(x, y), x^2 + y^2 < 1\}$$

b) Can you extend the result in (a) to the problem

$$\begin{aligned} Lu &= -F(x,y) \quad \text{in } D \\ u &= f(x,y) \quad \text{on } \partial D \end{aligned}$$

where

$$Lu = \frac{\partial}{\partial x}[A(x,y)\frac{\partial u}{\partial x} + B_1(x,y)\frac{\partial u}{\partial y}] + \frac{\partial}{\partial y}[B_2(x,y)\frac{\partial u}{\partial x} + C(x,y)\frac{\partial u}{\partial u}\partial y]$$

is an elliptic operator.

4. Find the solution of the initial value problem

$$u_t + u = \Delta u, \quad x \in \mathbb{R}^n, \ t > 0$$

$$u(x, 0) = h(x), \quad x \in \mathbb{R}^n$$

where h is continuous and bounded in $\mathbb{R}^n$ and Δ is the Laplace operator in $\mathbb{R}^n$.

PRELIMINARY EXAM PROBLEMS
Differential Equations (PDE), 2004/2

1. Solve the Cauchy problem
$$u_y = u_x^3, \quad u(x,0) = 2x^{3/2}.$$

2. (a) Verify, formally, that the PDE of the form
$$\Big\{ \frac{\partial}{\partial x}\big[F(x,y)\frac{\partial}{\partial x}\big] + \frac{\partial}{\partial y}\big[G(x,y)\frac{\partial}{\partial y}\big]\Big\}\Phi(x,y) = 0$$
has a solution of the type $\Phi(x,y) = X(x)Y(y)$, if $F(x,y)$ and $G(x,y)$ are "separable" in the variables, i.e. $F(x,y) = p(x)f(y), G(x,y) = q(x)w(y)$. Then write down the system of two ODE's for $X(x)$ and $Y(y)$.
(b) If $\Phi(0,y) = \Phi(1,y) = 0$ for all y, verify that the $x-$dependence of the problem in Part (a) is equivalent to the system
$$\frac{d}{dx}\Big[p(x)\frac{dX}{dx}\Big] + \lambda q(x)X = 0, \quad X(0) = X(1) = 0,$$
where $p(x)$ and $q(x)$ are real and positive with continuous derivatives in the interval $[0,1]$ and λ is constant.

3. Use Duhamel's principle to solve the IVP
$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = x_1 + x_2 + t,$$
$$u(x_1,x_2,x_3,0) = u_t(x_1,x_2,x_3,0) = 0.$$

4. Let u be a solution of IVP
$$u_t - ku_{xx} = 0, \quad x \in R, \quad t > 0,$$
$$u(x,0) = f(x).$$
where $f(x)$ is continuous on R. Assume that $u(x,t)$ tends to zero uniformly for $t > 0$ as $x \to \pm\infty$. Show that $|u(x,t)| \le M, x \in R, t > 0,$ if $|f(x)| \le M, x \in R$.

PRELIMINARY EXAM PROBLEMS
Differential Equations (PDE), 2005/2, 3 hours

1. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1,$$

$$u = y^4, \quad x^2 + y^2 = 1.$$

2. (a). Find, for all positive and negative values of a constant λ, the real solutions of the equation

$$\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial z}{\partial t}$$

that are of the form $z = \mathrm{e}^{\lambda t}\phi(x)$.

(b). If c is not an integer multiple of π, show that there exists a solution of this equation which remains finite as $t \to \infty$, which is zero when $x = 0$, and which assumes the value $\mathrm{e}^{-\mathrm{t}}$ when $x = 1$. Find this solution.

3. (a). Let $u(x, t)$ be a solution of the equation

$$u_t - k u_{xx} = F(x, t), \ k > 0, \tag{1}$$

for $\{(x, t) \mid 0 < x < L, \ t > 0\}$, where L is a fixed positive number, and $u(x, t)$ is continuous in $\{(x, t) \mid 0 \le x \le L, \ t \ge 0\}$. Prove that the maximum of $u(x, t)$ is attained at $t = 0$, or $x = 0$, or $x = L$, if $F(x, t)$ is negative valued in $\{(x, t) \mid 0 < x < L, t > 0\}$.

(b). Construct a counter example if $F(x, t)$ in (1) is positive in the region.

4. The function $\frac{-1}{2\pi} K_0(\alpha r)$ is a fundamental solution for the equation

$$\nabla^2 u - \alpha^2 u = 0 \quad in \quad \Omega,$$

where α is a constant, $\Omega \subset R^2$ and $K_0(\alpha r)$ is the zero order modified Bessel function of the second kind, r is the distance from a fixed point (ξ, η) to any point (x, y) in Ω.

Prove that the Green's function for the equation above defined by

$$\nabla^2 G - \alpha^2 G = \delta(x - \xi)\delta(y - \eta) \quad in \quad \Omega,$$

$$G = 0 \quad on \quad \partial\Omega,$$

is unique, where δ and ∇^2 denote the Dirac Delta and Laplace operators respectively.

PRELIMINARY EXAM PROBLEMS
Differential Equations (PDE), 3 hours, 13.09.2006

1. Let Ω denote the unbounded set $|x| > 1$. Function $u \in C^2(\bar{\Omega})$ satisfies $\Delta u = 0$ in Ω and $\lim_{x\to\infty} u(x) = 0$. Show that
$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$
Hint: Apply the maximum principle to a spherical shell.

2. (a). Solve the following problem
$$xu_x + yu_y = u + 1, \quad u|_\Gamma = x^2$$
if $\Gamma = \{(x, y) : y = x^2\}$.
(b). Use d'Alembert's formula to determine $u(1, 2)$ if
$$\begin{aligned} &u_{tt} - 4u_{xx} = 0, 0 < x < 2, t > 0, \\ &u(x, 0) = \sqrt{x}, \quad u_t(x, 0) = 2 - x, \quad 0 \le x \le 2, \\ &u(0, t) = 0, \quad u_t(2, t) = 0. \end{aligned}$$

3. Solve the following problem by Fourier's method.
$$\begin{aligned} &u_{tt} = u_{xx} + 2a, \quad 0 < x < l, \quad a - constant, \\ &u(0, t) = 0, \quad u(l, t) = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0. \end{aligned}$$

4. Consider the following initial value problem
$$\begin{aligned} u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) = x, \quad 0 \le x \le 1, \\ u(0, t) = \sin t, \quad u(1, t) = \cos t, \quad t \ge 0. \end{aligned}$$
Using the maximum principle, show that:
(a) $u(x, t) \le 1$ for $t \in [0, T]$ for every $T > 0$;
(b) the problem has at most one solution.

METU - Department of Mathematics
Graduate Preliminary Exam

Partial Differential Equations

Duration : 180 min. Fall 2008

1. Solve the following initial value problem.

$$u_x{}^2 - 3u_y{}^2 = u, \quad u(x,0) = x^2.$$

2. Let D be the region $(0,1) \times (0,1) \subset \mathbb{R}^2$ and let $u(x,y) \in C^2(D) \cap C^0(\overline{D})$ be a non-constant function. Set $M = \max(u)$ in $\overline{D}$.

   a) Show that if $u(x,y)$ solves the equation

$$\nabla^2 u(x,y) + a(x,y)u_x + b(x,y)u_y = F(x,y) \text{ in } D$$

   and if $F(x,y) > 0$ in D, then $u(x,y) < M$ for all $(x,y) \in D$.

   b) True or false ? The same conclusion holds if $u(x,y)$ solves the equation $u_{xy} = 0$ in D. (**Prove the statement or give a counter example**).

3. Consider the following Dirichlet problem.

$$\nabla^2 u = 0 \quad in \ \Omega = \{(r,\theta) : r > 1, 0 < \theta < \pi/2\}$$

$$u(r,0) = u(r,\pi/2) = 0 \text{ for } r \geq 1$$

$$u(1,\theta) = \sin(2\theta) \text{ for } 0 < \theta < \pi/2.$$

   a) Find the **bounded** solution of this problem.

   b) Find an unbounded solution, if there is any.

   c) Write a Neumann problem in Ω for which the function $u(r,\theta)$ of part (a) is a solution.

4. Let $G(x,t)$ be the heat kernel $G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

a) Show that $u(x,t) = 2\int_0^t G(x, t-t') f(t') dt'$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x,t) : x > 0,\ t > 0\}$$

$$u(x,0) = 0,\ x > 0.$$

Hint : Do **not** verify the uniform convergence of the integral, but indicate when you use this property.

b) Verify that $-\frac{\partial u}{\partial x}|_{x=0} = f(t),\ t > 0.$

Hint : In the integral, first make the change of variable given by $s^2 = \frac{x^2}{4(t-t')}$.

METU - Department of Mathematics
Graduate Preliminary Exam

Partial Differential Equations

February, 2009

Duration: 180 min.

1. Solve the Cauchy problem

$$u_y = u_x^3, \quad u(x,0) = 2x^{3/2}$$

2. The initial value problem

$$u_{tt} - c^2 u_{xx} = x^2, \quad t > 0, \quad x \in \mathbb{R}$$

$$u(x,0) = x, \quad u_t(x,0) = 0$$

is given.

a) First state the Cauchy-Kowalewski theorem for a linear equation and then decide whether the above problem has a unique solution or not.

b) Find a particular time-independent solution of the differential equation.

c) Find the solution of the given initial value problem.

3. a) State the uniqueness theorems for the solutions of Dirichlet and Neumann problems defined for the Laplace equation $\Delta u = 0$ in $\mathbb{R}^3$.

b) Show that the solutions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\Delta u - [3 - (x_1^2 + x_2^2 + x_3^2)]u = 0$$

in

$$\Omega := \{x \in \mathbb{R}^3 : |x_i| < 1, \ i = 1,2,3\}$$

for the Dirichlet and Neumann problems are unique.

4. Let u be a solution of IVP

$$u_t - k u_{xx} = 0, \quad x \in R, \quad t > 0,$$
$$u(x,0) = f(x).$$

where $f(x)$ is continuous on R. Assume that $u(x,t)$ tends to zero uniformly for $t > 0$ as $x \to \pm\infty$. Show that $|u(x,t) \leq M, x \in R, t > 0$, if $|f(x)| \leq M, x \in R$.

METU - Department of Mathematics
Graduate Preliminary Exam

Partial Differential Equations

Duration : 180 min. Fall 2010
Each question is 25 pts.

1. Determine if the following Cauchy problem has a solution in the neighbourhood of the point (1,0)

$$yz_x - xz_y = 0,$$

   a) $z = 2y$ and $x = 1$

   b) $z = 2y$ and $x = 1 + y$.

2. Let $\Omega = \{x \in \mathbb{R}^3 : \quad |x| > 1\}$. Let $u \in C^2(\bar{\Omega})$ and suppose that u satisfies the Laplace equation in Ω

$$\Delta u(x) = 0, \text{for all } x \in \Omega.$$

   Show that

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

   if $\lim_{x \to \infty} u(x) = 0$.

3. For the wave equation in $\mathbb{R}^3$

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

   find a general form of the plane wave solution. That is, a solution of the form $u(t, x, y, z) = v(t, s)$ where $s = \alpha x + \beta y + \gamma z$ and α, β, $\gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$.

4. Consider the P.D.E.

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = xe^{-y}. \quad (*)$$

a) Determine the type of this PDE and find the characteristic curves.

b) Consider the given PDE as a first order PDE for $q = \dfrac{\partial z}{\partial y}$.

Solve this first order PDE for q. Then find the general solution of (*).

c) Find two different solutions $z(x, y)$ which satisfy the condition

$$z(x, x) = 1 \text{ for } x \in \mathbb{R}.$$

d) True or false ? Explain.

There exists a solution $z(x, y)$ such that $z(x, x) = 1$, $\dfrac{\partial z}{\partial \mathbf{n}} = 0$ where $\mathbf{n}$ is the unit normal vector to the line $y = x$ in $\mathbb{R}^2$.

M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

Partial Differential Equations

Duration : 180 min.

Each question is 25 pt.

NOTATION :
∇, Δ denote the gradient and the Laplace operators, respectively.

1. **a.** For a linear second order differential equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

define the elliptic, parabolic and hyperbolic equation at point (x, y).

Consider the equation $u_{yy} - yu_{xx} = 0$.

b. Determine where the equation is elliptic, parabolic, hyperbolic.

c) Determine the characteristics in the region $\mathcal{H} = \{(x, y) : y > 0\}$.

2. **a.** Give the definition of Dirichlet and Neumann problems for Laplace equation in domain Ω.

b. Using the energy identity

$$\int_\Omega \left(\sum u_{x_i}^2\right) dx + \int_\Omega u\Delta u dx = \int_{\partial\Omega} u\frac{du}{dn} dS.$$

prove that if $u \in C^2(\bar{\Omega})$ is a solution of a Dirichlet problem in Ω, then it is unique.

c.) Explain if there exists a solution for each of the following problems in the unit disk $\Omega = \{(r, \theta) : r < 1\} \in \mathrm{R}^2$.

- $\Delta(u) = 0$, $u|_{\partial\Omega} = \cos(\theta)$.

- $\Delta(u) = 0$, $\frac{\partial u}{\partial n}|_{\partial\Omega} = \cos(\theta)$.

(Hint : For $f, g \in C^2(\Omega)$ one has the Stokes' formula

$$\int_\Omega (\nabla f . \nabla g) dA + \int_\Omega (f \Delta g) dA = \int_{\partial\Omega} (f \frac{\partial g}{\partial n}) dl.)$$

3. For the equation

$$\Delta u(x) + u(x) = 0 \quad x \in \mathbb{R}^3$$

find the spherically symmetric solution. That is a solution of the form $u = f(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

(Hint: in the resulting ODE for f introduce new variable $y(r) = rf(r)$).

4. For $T \geq 0$ let $Q_T = \{(x, t) : 0 < x < L, 0 < t \leq T\}$ and $B_T = \bar{Q}_T \backslash Q_T$ ($\bar{Q}_T$ denotes the closure of Q_T). Suppose that $u(x, t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x, t) u_{xx} - b(x, t) u_x - c(x, t) u < 0 \quad \text{on} \quad Q_T,$$

where $a(x, t) \geq 0$, $c(x, t) \leq 0$ in Q_T, and

$$u(x, t) \leq 0 \quad \text{on} \quad B_T.$$

Show that $u(x, t) \leq 0$ on $\bar{Q}_T$.

(Hint: show that $u(x, t)$ cannot have positive local maximum in Q_T.)

PRELIMINARY EXAM - Sep. 2012

Partial Differential Equations

Q.1	Q.2	Q.3	Q.4	Total

Duration : 3 hr.

Each question is 25 pt.

1. Consider the PDE

$$x^4(\frac{\partial z}{\partial x})^2 - yz\frac{\partial z}{\partial y} - z^2 = 0 \ \text{ in } \Omega = \{(x,y,z) \in \mathbf{R}^3 : x > 0,\ y > 0,\ z > 0\} \ \ (*).$$

a) Transform the given PDE by using the change of variables

$$X = \frac{1}{x},\ Y = \frac{1}{y},\ Z = \ln(z).$$

b) Solve the PDE you obtained in (a) to get the **complete integral** of the original PDE (*).

c) Find all singular solutions, if any, of (*).

2. Consider the problem

$$u_{xx} - u_t = tu \text{ for } (x,t) \in \mathbb{R} \times (0,\infty)$$
$$u(x,0) = \phi(x).$$

a) Determine the ODE satisfied by the function $f(t)$ if $u(x,t) = f(t)v(x,t)$ where $v(x,t)$ is the solution of the problem

$$v_{xx} - v_t = 0 \text{ for } (x,t) \in \mathbb{R} \times (0,\infty)$$
$$v(x,0) = \phi(x).$$

b) Determine $f(t)$.

c) Let $u_1(x,t),\ u_2(x,t)$ be the solutions of the given problem corresponding to the boundary conditions $\phi_1(x),\ \phi_2(x)$ respectively. Show that if

$$\sup_{\mathbb{R}} |\phi_1(x) - \phi_2(x)| \le 1$$

then $|u_1(x,t) - u_2(x,t)| \le e^{-t^2/2}$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$.

3. Consider the equation

$$U_{xy} + 2y\,U_{xx} + U_x = 0\ .$$

(A) Determine the type of this equation.

(B) Introduce a suitable system of coordinate in which the principal part of this equation assumes a canonical form.

(C) Find the general solution of the above equation.

(D) Find the solution of the above equation which satisfies

$$U_x(x,0) = x^2 \quad \text{and} \quad U(0,y) = 0\ .$$

(Hint : Notice that $M = x - H(y)$ is a solution of the equation

$$H'(y)\,M_x + M_y = 0).$$

4. Let $\Omega \subset \mathbb{R}^2$ be an open connected and bounded region and let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous functon which is C^2 on Ω. Show that if u satisfies

$$\nabla^2 u = u^3 \text{ in } \Omega$$

$$u|_{\partial\Omega} = 0$$

then $u \equiv 0$ in Ω.

M.E.T.U.
DEPARTMENT OF MATHEMATICS

Preliminary Exam - Feb. 2013
Partial Differential Equations

Duration : 3 hr.

Each question is 25 pt.

1. Consider the following problem

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy$$

$$u(1,y) = e^y.$$

a) Find the solution.

b) Discuss the existence and uniqueness of the solution in a neighbourhood of $(1, y_0)$ as y_0 varies.

2. Consider the following problem.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in } \Omega = \{(x,t) : 2 > x > 0, t > 0\}$$

$$u(0,t) = u(2,t) = 0, \;\; u(x,0) = 1 - |x-1| \text{ for } 2 \geq x \geq 0.$$

a) Solve this problem by the method of separation of variables.

(Write the integral expressions for the coefficients, but do not compute the integrals).

b) Using the heat kernel, write the integral form of the solution of the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in } \Omega = \{(x,t) : x \in \mathbb{R}, \;\; t > 0\}$$

$$u(x,0) = \begin{cases} 1 - |x-1| & \text{if } 2 \geq x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $|u(x,t)| \leq 1$ in $\mathbb{R} \times (0, \infty)$.

c) Can you obtain the solution in (a) by restricting the solution in (b) to $\Omega = \{(x,t) : 2 > x > 0,\ t > 0\}$?

3. Consider the following Monge-Ampere equation in two variables

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - (\frac{\partial^2 u}{\partial y \partial x})^2 = 1.$$

a) Determine the general solution $u(x,y)$ which satisfies

$$\frac{\partial^2 u}{\partial x^2} = 1 = \frac{\partial^2 u}{\partial y^2}.$$

b) Find two distinct solutions of the BVP : $u(x,y)|_{x^2+y^2=1} = 1.$

(Hint : Replace 1 by -1 in (a) and find the corresponding general solution).

c) Verify that the mean value property

$$u(x_0, y_0) = \frac{1}{2\pi r}\int_{S^1} u(x,y)ds$$

(where S^1 is the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$)

and the maximum principle do not hold for the solutions of the given equation.

4. Let $\Omega \subset \mathbb{R}^2$ be an open connected and bounded region, $\overline{\Omega}$ be the closure of Ω and $\partial\overline{\Omega}$ be the boundary. Let $f(x,y) \in C^2(\overline{\Omega})$ be a **subharmonic** function, that is f satisfies the inequality $\nabla^2 f \geq 0$ in Ω. Equivalently, for any $p \in \Omega$ and any disc $\overline{D}(p;r) \subset \overline{\Omega}$ one has

$$f(p) \leq \frac{1}{2\pi r}\int_{\partial D(p;r)} f ds.$$

a) Show that if $f(p) = \max_{\overline{\Omega}}(f)$ for some $p \in \Omega$, then f is constant.

b) Show that if $u(x,y)$ is harmonic in $\overline{\Omega}$ then $f(x,y) = |\nabla u(x,y)|^2$ is subharmonic.

c) Find a nonconstant subharmonic function f in $\overline{D}(0;1)$ such that $\max_{\overline{D}}(f) = 1$.

(Hint : You may use (b)).

Partial Differential Equations

Problem 1. Consider a partial differential equation with constant coefficients

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y) \qquad a \neq 0.$$

Given that the equation is parabolic.
a. Write the equation in canonical form using independent variables (a, b) (DO NOT evaluate the coefficients of canonical form). How variables (x, y) and (a, b) are related?
b. Show that the canonical form can be simplified to

$$v_{aa} + \bar{d}v_b = \bar{g}(a, b)$$

by the change of variable $u = ve^{\lambda a + \mu b}$.

Problem 2. Let D be a bounded domain in $\mathbb{R}^n$. Consider a region $\Omega = \{(x, t) : x \in D, \quad 0 < t \leq T\}$ and a function $u(x, t) \in C^2(\Omega) \cup C(\bar{\Omega})$. Show that if $u_t - a\Delta u > 0$ on Ω, where $a > 0$ then u can not assume a local minimum in Ω.

Problem 3. Consider the initial value problem

$$u_y = G(x, y, u, u_x) \qquad u(x, 0) = f(x),$$

where G is twice continuously differentiable and f is three times continuously differentiable. Show that the problem has a unique solution in the neighborhood of the initial curve.

Problem 4. Let u and v be two solutions of the one-dimentional wave equation

$$u_{tt} = c^2 u_{xx}.$$

Show that $\frac{d}{dt}\int_a^b u_t v_t + c^2 u_x v_x = 0$ provided that $u = 0$,
$v = 0$ for $x = a$, $t > 0$ and $x = b$, $t > 0$.

Problem 1.
Solve the problem

$$u_x + u_y = x \quad u(x,0) = h(x).$$

Problem 2.
For the wave equation in $\mathbb{R}^3$

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

find a general form of the plane wave solution. That is solution of the form $u(t,x,y,z) = v(t,s)$ where $s = \alpha x + \beta y + \gamma z$ and $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, $\alpha^2+\beta^2+\gamma^2 = 1$.
Problem 3.
Let $\Omega = \{x \in \mathbb{R}^3 : \quad |x| > 1\}$. Let $u \in C^2(\bar{\Omega})$ and u satisfies the Laplace equation in Ω, $\Delta u = 0 \quad x \in \Omega$. Show that

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

if $\lim\limits_{x\to\infty} u(x) = 0$.
Problem 4.
Show that the problem

$$\begin{aligned} &u_t = -u_{xx} \quad t > 0,\ -\infty < x < \infty \\ &u(x,0) = f(x) \end{aligned}$$

is not well posed. Hint: Consider solutions of the equation

$$u_t = -u_{xx} \quad t > 0,\ -\infty < x < \infty$$

$u_1(x,t) = 1$, and $u_2(x,t) = 1 + \frac{1}{n}e^{n^2 t}\sin(nx)$, $n = 1, 2, 3, \ldots$.

Partial Differential Equations

Problem 1. For the equation

$$(y^2 - x)z_x + yz_y = z$$

(a) give an example of the initial curve so that the corresponding Cauchy problem has a unique solution (do not solve the problem); (Explain)
(b) give an example of the initial curve so that the corresponding Cauchy problem has no solution. (Explain)

Problem 2. In the region $D := \{(x, y) : y > 1\}$ determine the type and transform to the canonical form the following equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = 0.$$

Problem 3.
(a) Show that if $u(x, t)$ satisfies

$$\begin{array}{ll} u_{tt} + u_t = u_{xx}, & 0 < x < L,\ t > 0 \\ u(0,t) = 0,\ u(L,t) = 0 & t \geq 0 \end{array}$$

then $E(t) = \int_0^L (u_t^2 + u_x^2)dx$ is a decreasing function.

(b) Use (a) to prove uniqueness of solution for the initial-boundary value problem

$$\begin{array}{ll} u_{tt} + u_t = u_{xx}, & 0 < x < L,\ t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 \leq x \leq L \\ u(0,t) = a(t),\ u(L,t) = b(t), & t \geq 0 \end{array}$$

Problem 4. Does the following problem has analytic solution in the neighborhood of the point $(1, 1)$

$$u_{tt} = u_{xx}u_x^2,\ u(x, 1) = \sin x,\ u_t(x, 1) = e^x\ ?$$

(Explaine).

TMS- PDE

Feb. 15, 2016

1) Consider the p.d.e. $x^4p^2 - yzq - z^2 = 0$, where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, in the region $\{(x, y, z) \in \mathbf{R}^3 : x > 0, y > 0, z > 0\}$.

a) Write the p.d.e. obtained by applying the transformation

$$X = \frac{1}{x}, Y = \frac{1}{y}, Z = \ln(z)$$

to the given equation.

b) Solve the p.d.e. you obtained in (a) to get the **complete integral of the original p.d.e**.

c) Find all singular solutions, if any, of the given equation.

2) Let r, θ denote the polar coordinates in the plane. Find **all** harmonic functions $u(r, \theta)$ in the region $\mathcal{R} = \{(r, \theta) : r > 1\}$, which satisfy $u(1, \theta) = 1 + \cos(\theta)$

(a) if u is bounded in $\mathcal{R}$.

(b) if u is unbounded in $\mathcal{R}$.

3) Consider the p.d.e. $x^2\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} = 0.$

a) Determine the regions in $\mathbf{R}^2$ in which the given equation is (i) elliptic, (ii) parabolic, (iii) hyperbolic.

b) Find the normal form of this equation in the region $\{(x, y) : x > 0\}$.

c) Using the normal form, find the general solution of the given equation for $x > 0$.

4) Solve the boundary value problem

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ in $\{(x, y) \in \mathbf{R}^2 : \pi > x > 0, b \geq t > 0\}$,

$u(0, t) = u(\pi, t) = 0,\ u(x, 0) = 1 - \cos(4x).$

TMS. Differential Equations (PDE)

1. (a) Find the general solution of the equation

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -xy \tag{1}$$

   (b) Determine the solution of (1) passing through the curve $y = x^2, z = x^3$.

2. Reduce the equation $yu_{xx} + xu_{yy} = 0$ to the canonical forms in the plane.

3. Suppose $u(x,t)$ is the solution to

$$\begin{array}{ll} u_t - u_{xx} = x, & 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0 & t \geq 0 \\ u(x,0) = 0 & 0 \leq x \leq 1. \end{array}$$

   Apply the maximum principle to show that $u(x,t) \leq \dfrac{x - x^3}{6}$ for $0 < x < 1$ and $t > 0$.

4. Find a harmonic function $u(r,\theta)$ in the annulus $2 < r < 4$ with $u(2,\theta) = 1$ and $u(4,\theta) = \sin^2 2\theta$.

Partial Differential Equations

Problem 1. Find solution of the equation

$$xyz_x + xzz_y = yz$$

passing through the curve $z = 1 + y^2$, $x = 1$ (if exists).

Problem 2. Using Green's first identity

$$\iint_{\partial D} v \frac{\partial u}{\partial n}\, dS = \iiint_D \nabla v \cdot \nabla u \, dx + \iiint_D v \Delta u \, dx,$$

prove the uniqueness of solution for the Robin problem

$$\Delta u = 0 \quad \text{in} \quad D, \qquad \frac{\partial}{\partial n} u(x) + a(x)u(x) = h(x) \quad \text{on} \quad \partial D,$$

provided $a(x) > 0$ on ∂D.

Problem 3. Solve the Dirichlet problem for the exterior of a circle

$$\begin{array}{ll} \Delta u = 0, & x^2 + y^2 > a^2 \\ u = h, & x^2 + y^2 = a^2 \end{array}$$

and u bounded as $x^2 + y^2 \to \infty$, given that $h(a, \theta) = \sum_{n=1}^{\infty} \alpha_n \cos 2n\theta$ (in polar coordinates (r, θ)).

Problem 4. Let $\Omega = \{(x, t) : x \in (0, a),\ t \in (0, T]\}$ and $\partial_p \Omega = \{(x, t) : x = 0, t \in [0, T] \,\text{or}\, x = a, t \in [0, T] \,\text{or}\, x \in [0, a], t = 0\}$. Suppose $u \in C^2(\Omega) \cup C(\bar{\Omega})$ satisfies $u_t - u_{xx} + cu < 0$, where $c \geq 0$, in Ω. Show that $\max_{\bar{\Omega}} u = \max_{\partial_p \Omega} u$ given that $\max_{\partial_p \Omega} u > 0$.