16 February 2004

Graduate Preliminary Examination
Real Analysis
Duration: 3 hours

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L_1(\mu)$ with $f(x) > 0$ a.e. Prove that if A is a measurable set such that $\int_A f d\mu = 0$, then $\mu(A) = 0$,

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and (a, b) be a finite, non-empty interval in $\mathbb{R}$. Let $f : X \times (a, b) \to \mathbb{R}$ satisfy

a) $F(t) = \int f(x, t)dx$ is defined $\forall t \in (a, b)$

b) $\frac{\partial f}{\partial t}$ is defined everywhere in $X \times (a, b)$

c) There is an integrable $g : X \to [0, \infty)$ such that $|\frac{\partial f}{\partial t}(x, t)| \le g(x)$ $\forall x \in X$, $t \in (a, b)$.

Prove that both $F'(t)$ and $\int \frac{\partial f}{\partial t}(x, t)dx$ exist $\forall t \in (a, b)$ and are equal.

3. Let $0 < p < q < \infty$ and $(X, \mathcal{M}, \mu)$ be a measure space. Prove that

a) $L^p \not\subset L^q \Leftrightarrow X$ contains sets of arbitrarily small positive measure,

but

b) $\ell_p \subsetneqq \ell_q$.

4.

a) State the Lebesgue-Radon-Nikodym Theorem for signed measures.

b) For $j = 1, 2$ let μ_j, ν_j be σ-finite measures on $(X, \mathcal{M}_j)$ s.t. $\nu_j << \mu_j$ (ν_j is absolutely continuous with respect to μ_j). Prove that $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and $\dfrac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \dfrac{d\nu_1}{d\mu_1}(x_1) \cdot \dfrac{\partial \nu_2}{\partial \mu_2}(x_2)$.

METU - Department of Mathematics
Graduate Preliminary Exam

Real Analysis

February, 2009

Duration: 180 min.

1. Characterize metric spaces X for which the open sets form a σ-algebra.

2. a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere then f is a Lebesgue measurable function.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the derivative f' of f is a Lebesgue measurable function.

3. Construct an example of a subset $D \subseteq \mathbb{R}^2$ satisfying the following properties:

(i) the Lebesgue measure of $D \cap U$ is zero for every open set $U \subseteq \mathbb{R}^2$;

(ii) the set $D \cap U$ is of the second category for every nonempty open set $U \subseteq \mathbb{R}^2$;

(iii) for every point $(x_0, y_0) \in D$ the line $y = y_0 + x_0 - x$ is a subset of D.

4. a) State the Radon - Nikodym theorem for the interval $[0, 1]$ with the Lebesgue measure.

b) State the Hahn decomposition theorem.

c) Derive the Hahn decomposition theorem for the interval $[0, 1]$ from the Radon - Nikodym theorem and the Jordan decomposition theorem.

M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

REAL ANALYSIS

1. a) Let $\{f_n\}$ be a sequence of measurable functions on a measure space (X, S, μ) such that $\{f_n(x)\}$ is a bounded sequence for each $x \in X$. Show that the set

$$E = \{x \in X : \lim f_n(x) exists\}$$

is a measurable set.

b) Let (X, S, μ) be a measure space. Assume that $f : X \longrightarrow \mathbb{R}$ is a measurable function and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Show that the composition function gof is a measurable function.

2. a) Let $(X, \sum, \mu)$, (Y, Λ, ν) be measure spaces and $\mathcal{A}$ be algebra of subsets of $X \times Y$ generated by rectangles $A \times B$, $A \in \sum$, $B \in \Lambda$. By using of the Monotone Convergence Theorem show that the following function

$$\mu \times \nu : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+ \text{ defined by } \mu \times \nu(A \times B) := \mu(A) \cdot \nu(B)$$

is a pre-measure.

b) Let $A = ((a_{ij}))_{ij \in \mathbb{N}}$ be an infinite matrix of real numbers. Suppose $\lim\limits_{i \to \infty} a_{ij} = a_j \in \mathbb{R}$ and $\sup\limits_i |a_{ij}| = b_j$ with $\sum\limits_{j=1}^{\infty} b_j < \infty$. By application of the Dominated Convergence Theorem show that $\lim\limits_{i \to \infty} \sum\limits_{j=1}^{\infty} |a_{ij} - a_j| = 0$

3. a) Formulate Fubini's theorem.

b) Show that if $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, with $f(0, 0) = 0$, then

$$\int_0^1 [\int_0^1 f(x, y) dx] dy = -\frac{\pi}{4}, \quad \int_0^1 [\int_0^1 f(x, y) dy] dx = \frac{\pi}{4}$$

c) Can f above be integrable on $[0, 1] \times [0, 1]$? Explain.

4. Let $a \in \mathbb{R}$ and $K = \{f \in C^2[0\ 1] : f(0) = f(1) = 0,\ f'(0) = a\}$.

   Find $\min\limits_{f \in k} \displaystyle\int_0^1 (f''(x))^2 dx$ and a function $f \in K$ on which the minimum is attained.

   [Hint: apply Cauchy-Schwartz inequality to functions $\varphi(x) = f''(x), \psi(x) = 1 - x$]

METU MATHEMATICS DEPARTMENT

REAL ANALYSIS

FEBRUARY 2013 - TMS EXAM

Last Name: Signature:

Name:

1.) Let (X, S, μ) be a measure space, T be a metric space. Let $f : X \times T \to \mathbb{R}$ be a function. Assume that $f(\cdot, t)$ is measurable for each $t \in T$ and $f(x, \cdot)$ is continuous for each $x \in X$. Prove that if there exists an integrable function g such that for each $t \in T$, $|f(t,x)| \leq g(x)$ for $a \cdot a \cdot x$, then $F : T \to \mathbb{R}, F(t) = \int f(x,t) d\mu(x)$ is continuous.

2.) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and positive. Consider the set of all points in the upper half-plane being below the graph of f : $A_f = \{(x, y) \in \mathbb{R}^2 : 0 \leq y < f(x)\}$

Show that A_f is $\lambda \times \lambda$ - measurable and $(\lambda \times \lambda)(A) = \int f(x)dx$.

3.) For a function $f \in L_1(\mu) \cap L_2(\mu)$ establish the following properties:

a) $f \in L_p(\mu)$ for each $1 \leq p \leq 2$; and

b) $\lim\limits_{p \to 1^+} \| f \|_p = \| f \|_1$

4.) If $\{f_n\}$ is a norm bounded sequence of $L_2(\mu)$ then show that $\dfrac{f_n}{n} \to 0$ a.e. holds.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
FEBRUARY 2014 - TMS EXAM

1.

a) State and prove Fatou's Lemma.

b) Show that Fatou's Lemma may not be true, even in the presence of uniform convergence.

(Hint: You may find $f_n(x) = -\frac{1}{n}\chi_{[0,n]}$ on $\mathbb{R}$ useful).

2. Let E be a measurable set of finite measure; (f_n) be a sequence of measurable real valued fuction on E. Show that for given $\epsilon > 0$ and $\delta > 0$ $\exists$ measurable A in E with measure $m(A) < \delta$ and a natural number N such that $\forall x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

3.

a) Let $(X, \wedge, \mu)$ be a finite measure space. Let (f_n), (g_n) be two sequences of measurable functions and $f_n \to f$ in measure μ and $g_n \to g$ in measure μ. Show that $f_n g_n \to fg$ in measure.

b) By considering $f_n(X) = \sqrt{x^4 + \frac{x}{n}}$ and $f(x) = x^2$ on $(0, \infty)$ with Lebesgue measure, show that the conclusion may fail if the space has no finite measure.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that $\lim_{t\to\infty} \int f(x)\cos(xt)d\lambda(x) = 0$ when λ is the Lebesgue measure.

Preliminary Exam - February, 2015

Real Analysis

1) Let (X, M, μ) be a measure space, $g : X \rightarrow: [0, \infty]$ be a non-negative μ- measurable function. For each $E \in M$ define $\nu(E) = \int g\chi_E d\mu$.

a) Show that ν is a measure on (X, M).

b) Show that if f is any non-negative μ- measurable function then $\int f d\nu = \int fg d\mu$.

2) Let (X, M, μ) be a finite measure space, E_k be a sequence of sets in μ such that $\mu(E_k) > 1/100 \quad \forall k$. Let F be the set of points $x \in X$ which belong to infinitely many of these sets,E_k.

a) Show that $E \in M$.

b) Show that $\mu(E) \geq 1/100$

c) Show that conclusion (b) may fail if $\mu(X) = \infty$.

3) a) State the Dominated Convergence Theorem.

b) Let μ be a measure on the Borel subsets of $\mathbb{R}$, and $f \in L^1(\mu)$. Prove that the function $F(x) = \displaystyle\int_{(-\infty,x]} f d\mu$ is continous from the left.

c) Show that if $x \in \mathbb{R}$ and $\mu(x) = 0$ then F is continuous from the right at x.

4) Let μ, ν be finite measures on (X, M) and $\nu = \nu_1 + \nu_2$ be the Jordan decomposition of ν so that $\nu_1 \perp \mu$ and $\nu_2 << \mu$. Let $\lambda = \nu + \mu$.

a) Show that if A,B is a Hahn Decomposition for ν_1, μ then it is also a Hahn Decomposition for ν_1, ν_2.

b) Show that $\nu << \lambda$

c) Let $f = \frac{d\nu}{d\lambda}$. Show that $0 \leq f \leq 1 \quad \lambda - a.e.$ and the two sets $f^{-1}(\{1\}), f^{-1}([0, 1))$ form a Hahn Decomposition for ν_1, μ.

5) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $\int_{-\infty}^{\infty} f dx$ converges in the usual Riemann sense, let m denote the Lebesgue measure on $\mathbb{R}$.

a) Show that if $f(x) \geq 0$ m-a.e then $f \in L^1(m)$.

b) Give an example showing that the non-negativity assumption in part (a) is necessary.

Real Analysis

TMS

Feb. 2016

1) Let $(X, \sum, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be measurable. For $E \in \sum$ define $\nu(E) = \int_E f d\mu$ Show that ν is a measure (You will need a convergence theorem for countable additivity)

2) Let $f_n(x) = \dfrac{n^{3/2}x}{1+n^2x^2}$ Show that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$

3) Let (f_n) be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then prove that $\int |f_n - f| \to 0 \Leftrightarrow \int |f_n| \to \int |f|$.

4) Let h and g be integrable functions on (X, μ) and (Y, ν), and define $f(x, y) = h(x)g(y)$ Then show that f is integrable on $X \times Y$ and

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X h d\mu \int_Y g d\nu$$

Hint: Take $h = X_A$, $g = X_B$ where $A \subset X, B \subset Y$ are measurable sets.

Preliminary Exam - February, 2017

Real Analysis

1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(E) : E \in \mathcal{M}, A \subset E\}$. Prove that

a) μ^* is an outer measure on X.

b) $\forall A \in \mathcal{P}(X)\ \exists E \in \mathcal{M}$ such that $A \subset E$ and $\mu^*(A) = \mu(E)$.

2) Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0, 1]$ such that $\lim_{n\to\infty} \int_0^1 |f_n| dm = 0$ and there is an integrable function g on $[0, 1]$ such that $|f_n|^2 \leq g$, for each n.

a) Prove that $\lim_{n\to\infty} \int_0^1 |f_n|^2 dm = 0$

b) Prove that if $\lim_n f_n = f$ exists a.e. then f integrable on $[0, 1]$ and $\int f dm = 0$

3) If f is a complex valued measurable function on $(X, \mathcal{M}, \mu)$, define

$$R_f = \{z : \mu(\{x : |f(x) - z| < \epsilon\}) > 0\ \forall \epsilon > 0\}$$

Show that

a) R_f is closed. b) If $f \in L^\infty$ then R_f is compact.

4) Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space and define ν on $\mathcal{M}$ by $\nu(A) = 0$ if $\mu(A) = 0$; and $\nu(A) = \infty$ if $\mu(A) > 0$.

a) Show that ν is a measure on X and $\nu << \mu$. b) Find $\frac{d\nu}{d\mu}$.

GRADUATE PRELIMINARY EXAMINATION
ANALYSIS I (REAL ANALYSIS)
Fall 2005
September 12^{th}, 2005

Duration: 3 hours

1. Let (X, S, μ) be a measure space, T be a metric space. Let $f : X \times T \to \mathbf{R}$ be a function. Assume that $f(\cdot, t)$ is measurable for each $t \in T$ and $f(x, \cdot)$ is continuous for each $x \in X$. Prove that if there exists an integrable function g such that for each $t \in T$, $|f(x,t)| \leq g(x)$ for $a.a.x$, then $F : T \to \mathbf{R}$, $F(t) = \int f(x,t)d\mu(x)$ is continuous.

2. Let $\mathcal{G}$ be a set of half-open intervals in $\mathbf{R}$. Prove that $\cup_{G\in\mathcal{G}} G$ is Lebesgue measurable.

3. a) Let $f_n = \sin n^2 x \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $f_n \to 0$ weakly, but $f_n \not\to 0$ in measure.

b) Let $g_n = n^2 \chi_{[0,\frac{1}{n}]} \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $g_n \to 0$ in measure, but $g_n \not\to 0$ weakly.

c) Let A_n be a measurable subset of $[0,1]$ for each $n, \chi_{A_n} \in L_1$, and $\chi_{A_n} \to f$ weakly in L_1. Show that f is not necessarily a characteristic function of some measurable set.

4. Let $f : \mathbf{R} \to \mathbf{R}$. If $f \in L_1(m) \cap L_2(m)$ where m denotes the Lebesgue measure, prove that

a) $f \in L_p(m) \;\; \forall \;\; 1 \leq p \leq 2$

b) $\lim\limits_{p \to 1^+} ||f||_p = ||f||_1$.

TMS

Spring 2010

Real Analysis

1. a) Show that $f(x) = \dfrac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and $\int f d\mu = 1$

   b) A set E in $\mathbb{R}$ is said to be **null** if for any $\epsilon > 0$ we can cover E with countably many open intervals the sum of whose lengths is less than ϵ, i.e., $E \subset \cup_{n=1}^{\infty}(a_n, b_n)$ and $\displaystyle\sum_{1}^{\infty}(b_n - a_n) < \epsilon$.

   Show that any countable set in $\mathbb{R}$ is **null**.

2. Using Lebesgue Dominated Covergence Theorem, compute

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-kn^2}$$

   Hint: Consider $\mathbb{N}$ with the counting measure. Let

$$f_k : \mathbb{N} \to [0, \infty) \quad \text{be defined as} \quad f_k(n) = e^{-kn^2}. \quad \text{Use LDCT.}$$

3. a) Suppose $(f_n) \to f$ in measure and $(g_n) \to g$ in measure. Show $(f_n + g_n) \to f + g$ in measure.

   b) Let (f_n), (g_n) be sequences of measurable functions such that $(f_n) \to f$ in measure, $(g_n) \to g$ in measure and $f_n = g_n$ a.e. for every n. Show that $f = g$ a.e.

4. State Egoroff's theorem. Prove that in Egoroff's theorem the hypothesis $\mu(X) < \infty$ can be replaced by $|f_n| \leq g$ for all n where $g \in L^1(\mu)$

TMS

September 2011

Real Analysis

I. a) Let A_n be a sequence of measurable sets with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(\overline{lim} A_n) = 0$

Hint: $\overline{lim} A_n = \bigcap_{k=1}^{\infty} \cup_{n \geq k} A_n$

b) Let $f \in L_p(\mu)$ and $\epsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \quad \leq \quad \epsilon^{-p} \int |f|^p d\mu$$

II. a) Show that $f(x) = \dfrac{1}{\sqrt{x}}$ is Lebebsgue integrable over $[0, 1]$.

b) Compute $\lim_n \int_0^1 \frac{n \sin x}{1+n^2\sqrt{x}} dx$ and justify your calculations.

III. Assume $\mu(X) < \infty$. If f_n is a sequence of measurable functions on X such that $f_n \to f$ a.e. then prove that $f_n \to f$ [meas] also holds.

State the theorem(s) you used.

IV. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(x) \leq g(x)$ holds for all $x \in [a, b]$. Set $A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b]$ and $f(x) \leq y \leq g(x)\}$.

a) Show that A is a closed set (and hence a measurable subset of $\mathbb{R}^2$)

b) If $h : A \to \mathbb{R}$ is a continuous function, then show that h is Lebesque integrable over A and that

$$\int_A h d\lambda = \int_a^b \left(\int_{f(x)}^{g(x)} h(x, y) dy \right) dx$$

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2012 - TMS EXAM

1. Prove disprove:

a) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable then the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\ dm(x)$ is convergent.

b) If $\int_{-\infty}^{\infty} f(x)\ dm(x)$ is convergent then $f \in L^1$.

2. Compute $\lim_{n\to\infty} \displaystyle\sum_{k=0}^{\infty} \left(\frac{n}{2n+k}\right)^k$

(Hint: Use a convergence theorem)

3. Let $E \subset [0,1] \times [0,1]$ have the property that every horizontal section Ey is countable and every vertical section Ex has countable complement $[0,1] \setminus E_x$. Prove that E is not L-measurable.

4. Let (X, σ, μ) be a measure space.

a) Define convergence in measure

b) Let $\phi : \mathbb{C} \to \mathbb{C}$ be uniformly continuous. Let $f_n, f : X \to C$ be measurable and $f_n \to f$ in measure.

Show that $\phi \circ f_n$ converges to $\phi \circ f$ in measure.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2013 - TMS EXAM

1. (35 pts.) Denote by χ_A the characteristic function of $A \subseteq [0,1]$

a) Prove that $\psi(t,x) := (t, \frac{x+\chi_A(t)}{2})$ is measurable if and only if A is measurable

b) Suppose A is measurable, calculate the integral $\int_{[0,1]\times[0,1]} \psi d\mu$ where μ is the Lebesgue measure on $[0,1] \times [0,1]$

c) Give an example of $A \subseteq [0,1]$ which is not Lebesgue measurable.

2. (20 pts.) Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma, the monotone and the dominated convergence theorems as statements about infinite series.

3. (25 pts.) a) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which maps a Lebesgue measurable set onto a non-Lebesgue measurable set.

b) Why the condinition $|f_n| \leq g \in L_1$ in the Dominated convergence theorem cannot be replaced by $|f_n(t)| \leq M \in \mathbb{R}^+$.

4. (20 pts.) Given the counting measure ν on $P(\mathbb{R})$ and the Lebesgue measure μ on the Lebesgue algebra $\sum(\mathbb{R})$.

a) Show that μ is absolutely continuous with respect to ν.

b) Explain why the Radon-Nikodym theorem is not applicable to measures ν and μ.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2014 - TMS EXAM

1.

(a) State the Lebesgue Dominated Convergence Theorem.

(b) Use (a) to evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{dx}{\cos(x+\frac{1}{n})\, x^{\frac{1}{n}}}$$

where dx denotes integration with respect to Lebesgue measure.

[Be sure to explain why the hypotheses are satisfied when you quote (a).]

2. Either prove or provide an explicit counterexample to each of the following assertions: (you may quote without proof familiar relations and containments between L^p-spaces)

(a) If $f, g \in L^2([0,1])$ then $fg \in L^2([0,1])$. (Lebesgue measure)

(b) If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$. (Lebesgue measure)

(c) If $f, g \in L^2(\mathbb{R})$ then $fg \in \ell^2$. (counting measure)

3. Let λ denote Lebesgue measure on the real line.

(a) Prove that there is an open set $\mathcal{O}$ that is dense in $\mathbb{R}$ with $\lambda(\mathcal{O}) < 1$.

(b) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is uncountable.

(c) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is not compact.

4. Let m be Lebesgue measure on $[0,1]$ and n be counting measure and $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

(a) Show $\int\int f(x,y)dm(x)dn(y) \neq \int\int f(x,y)dn(y)dm(x)$.

(b) State the Fubini-Tonelli Theorem and state why the above result does not contradict the Theorem.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2015 - TMS EXAM

1. Formulate the Egoroff theorem (= the third Littlewood principle) and show that it fails in every measure space with infinite σ-finite measure.
Hint: Consider $f_n = \chi_{[n,n+1]}$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and suppose $X = \bigcup_n X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of measurable subsets of X. Use the monotone convergence theorem and the linearity of the integral to prove that, if f is a non-negative measurable real-valued function on X,
$$\int_X f d\mu = \sum_n \int_{X_n} f d\mu$$

Hint: Let $f_n = \displaystyle\sum_{k=1}^{n} f\ \chi_{X_k} = f\chi_{\cup_1^n X_k}$

3. Evaluate $\displaystyle\lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-kn^2}$ and prove your answer by using a measure theory theorem.

Hint: Let $f_k : \mathbb{N} \to [0, \infty)$ be defined by $f_k(n) = e^{-kn^2}, n \in \mathbb{N}$.

4. Using the Fubini/Tonelli theorems to justify all steps, evaluate the integral
$$\int_0^1 \int_y^1 x^{-3/2} \cos(\frac{\pi y}{2x}) dx dy$$

Hint: Consider $\int\int\ |x^{-3/2} \cos(\frac{\pi y}{2x})| dy dx$ and apply to Tonelli's theorem.

METU - Department of Mathematics
Graduate Preliminary Exam

Real Analysis

February, 2009

Duration: 180 min.

1. Characterize metric spaces X for which the open sets form a σ-algebra.

2. a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere then f is a Lebesgue measurable function.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the derivative f' of f is a Lebesgue measurable function.

3. Construct an example of a subset $D \subseteq \mathbb{R}^2$ satisfying the following properties:

(i) the Lebesgue measure of $D \cap U$ is zero for every open set $U \subseteq \mathbb{R}^2$;

(ii) the set $D \cap U$ is of the second category for every nonempty open set $U \subseteq \mathbb{R}^2$;

(iii) for every point $(x_0, y_0) \in D$ the line $y = y_0 + x_0 - x$ is a subset of D.

4. a) State the Radon - Nikodym theorem for the interval $[0, 1]$ with the Lebesgue measure.

b) State the Hahn decomposition theorem.

c) Derive the Hahn decomposition theorem for the interval $[0, 1]$ from the Radon - Nikodym theorem and the Jordan decomposition theorem.

M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

REAL ANALYSIS

1. a) Let $\{f_n\}$ be a sequence of measurable functions on a measure space (X, S, μ) such that $\{f_n(x)\}$ is a bounded sequence for each $x \in X$. Show that the set

$$E = \{x \in X : \lim f_n(x) exists\}$$

is a measurable set.

b) Let (X, S, μ) be a measure space. Assume that $f : X \longrightarrow \mathbb{R}$ is a measurable function and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Show that the composition function gof is a measurable function.

2. a) Let $(X, \sum, \mu)$, (Y, Λ, ν) be measure spaces and $\mathcal{A}$ be algebra of subsets of $X \times Y$ generated by rectangles $A \times B$, $A \in \sum$, $B \in \Lambda$. By using of the Monotone Convergence Theorem show that the following function

$$\mu \times \nu : \mathcal{A} \to \ \overline{\mathbb{R}}_+ \ \text{ defined by } \ \mu \times \nu(A \times B) := \mu(A) \cdot \nu(B)$$

is a pre-measure.

b) Let $A = ((a_{ij}))_{ij \in \mathbb{N}}$ be an infinite matrix of real numbers. Suppose $\lim\limits_{i \to \infty} a_{ij} = a_j \in \mathbb{R}$ and $\sup\limits_i |a_{ij}| = b_j$ with $\sum\limits_{j=1}^{\infty} b_j < \infty$. By application of the Dominated Convergence Theorem show that $\lim\limits_{i \to \infty} \sum\limits_{j=1}^{\infty} |a_{ij} - a_j| = 0$

3. a) Formulate Fubini's theorem.

b) Show that if $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, with $f(0, 0) = 0$, then

$$\int_0^1 [\int_0^1 f(x, y)dx]dy = -\frac{\pi}{4}, \quad \int_0^1 [\int_0^1 f(x, y)dy]dx = \frac{\pi}{4}$$

c) Can f above be integrable on $[0, 1] \times [0, 1]$? Explain.

4. Let $a \in \mathbb{R}$ and $K = \{f \in C^2[0\ 1] : f(0) = f(1) = 0,\ f'(0) = a\}$.

   Find $\min\limits_{f \in k} \displaystyle\int_0^1 (f''(x))^2 dx$ and a function $f \in K$ on which the minimum is attained.

   [Hint: apply Cauchy-Schwartz inequality to functions $\varphi(x) = f''(x), \psi(x) = 1 - x$]

METU MATHEMATICS DEPARTMENT

REAL ANALYSIS

FEBRUARY 2013 - TMS EXAM

Last Name: Signature:

Name:

1.) Let (X, S, μ) be a measure space, T be a metric space. Let $f : X \times T \to \mathbb{R}$ be a function. Assume that $f(\cdot, t)$ is measurable for each $t \in T$ and $f(x, \cdot)$ is continuous for each $x \in X$. Prove that if there exists an integrable function g such that for each $t \in T$, $|f(t,x)| \leq g(x)$ for $a \cdot a \cdot x$, then $F : T \to \mathbb{R}, F(t) = \int f(x,t)d\mu(x)$ is continuous.

2.) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and positive. Consider the set of all points in the upper half-plane being below the graph of f : $A_f = \{(x, y) \in \mathbb{R}^2 : 0 \leq y < f(x)\}$

Show that A_f is $\lambda \times \lambda$ - measurable and $(\lambda \times \lambda)(A) = \int f(x)dx$.

3.) For a function $f \in L_1(\mu) \cap L_2(\mu)$ establish the following properties:

a) $f \in L_p(\mu)$ for each $1 \leq p \leq 2$; and

b) $\lim\limits_{p \to 1^+} \| f \|_p = \| f \|_1$

4.) If $\{f_n\}$ is a norm bounded sequence of $L_2(\mu)$ then show that $\dfrac{f_n}{n} \to 0$ a.e. holds.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
FEBRUARY 2014 - TMS EXAM

1.

a) State and prove Fatou's Lemma.

b) Show that Fatou's Lemma may not be true, even in the presence of uniform convergence.

(Hint: You may find $f_n(x) = -\frac{1}{n}\chi_{[0,n]}$ on $\mathbb{R}$ useful).

2. Let E be a measurable set of finite measure; (f_n) be a sequence of measurable real valued fuction on E. Show that for given $\epsilon > 0$ and $\delta > 0$ $\exists$ measurable A in E with measure $m(A) < \delta$ and a natural number N such that $\forall x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

3.

a) Let $(X, \wedge, \mu)$ be a finite measure space. Let (f_n), (g_n) be two sequences of measurable functions and $f_n \to f$ in measure μ and $g_n \to g$ in measure μ. Show that $f_n g_n \to fg$ in measure.

b) By considering $f_n(X) = \sqrt{x^4 + \frac{x}{n}}$ and $f(x) = x^2$ on $(0, \infty)$ with Lebesgue measure, show that the conclusion may fail if the space has no finite measure.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that

$\lim_{t\to\infty} \int f(x)\cos(xt)d\lambda(x) = 0$ when λ is the Lebesgue measure.

Preliminary Exam - February, 2015

Real Analysis

1) Let (X, M, μ) be a measure space, $g : X \to : [0, \infty]$ be a non-negative μ- measurable function. For each $E \in M$ define $\nu(E) = \int g\chi_E d\mu$.

a) Show that ν is a measure on (X, M).

b) Show that if f is any non-negative μ- measurable function then $\int f d\nu = \int fg d\mu$.

2) Let (X, M, μ) be a finite measure space, E_k be a sequence of sets in μ such that $\mu(E_k) > 1/100 \quad \forall k$. Let F be the set of points $x \in X$ which belong to infinitely many of these sets,E_k.

a) Show that $E \in M$.

b) Show that $\mu(E) \geq 1/100$

c) Show that conclusion (b) may fail if $\mu(X) = \infty$.

3) a) State the Dominated Convergence Theorem.

b) Let μ be a measure on the Borel subsets of $\mathbb{R}$, and $f \in L^1(\mu)$. Prove that the function $F(x) = \displaystyle\int_{(-\infty, x]} f d\mu$ is continous from the left.

c) Show that if $x \in \mathbb{R}$ and $\mu(x) = 0$ then F is continuous from the right at x.

4) Let μ, ν be finite measures on (X, M) and $\nu = \nu_1 + \nu_2$ be the Jordan decomposition of ν so that $\nu_1 \perp \mu$ and $\nu_2 << \mu$. Let $\lambda = \nu + \mu$.

a) Show that if A,B is a Hahn Decomposition for ν_1, μ then it is also a Hahn Decomposition for ν_1, ν_2.

b) Show that $\nu << \lambda$

c) Let $f = \frac{d\nu}{d\lambda}$. Show that $0 \leq f \leq 1 \quad \lambda - a.e.$ and the two sets $f^{-1}(\{1\}), f^{-1}([0, 1))$ form a Hahn Decomposition for ν_1, μ.

5) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\int_{-\infty}^{\infty} f dx$ converges in the usual Riemann sense, let m denote the Lebesgue measure on $\mathbb{R}$.

a) Show that if $f(x) \geq 0$ m-a.e then $f \in L^1(m)$.

b) Give an example showing that the non-negativity assumption in part (a) is necessary.

Real Analysis

TMS

Feb. 2016

1) Let $(X, \sum, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be measurable. For $E \in \sum$ define $\nu(E) = \int_E f d\mu$ Show that ν is a measure (You will need a convergence theorem for countable additivity)

2) Let $f_n(x) = \dfrac{n^{3/2}x}{1+n^2x^2}$ Show that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$

3) Let (f_n) be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then prove that $\int |f_n - f| \to 0 \Leftrightarrow \int |f_n| \to \int |f|$.

4) Let h and g be integrable functions on (X, μ) and (Y, ν), and define $f(x, y) = h(x)g(y)$ Then show that f is integrable on $X \times Y$ and

$$\int_{X\times Y} fd(\mu \times \nu) = \int_X hd\mu \int_Y gd\nu$$

Hint: Take $h = X_A$, $g = X_B$ where $A \subset X, B \subset Y$ are measurable sets.

Preliminary Exam - February, 2017

Real Analysis

1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(E) : E \in \mathcal{M}, A \subset E\}$. Prove that

a) μ^* is an outer measure on X.

b) $\forall A \in \mathcal{P}(X)\ \exists E \in \mathcal{M}$ such that $A \subset E$ and $\mu^*(A) = \mu(E)$.

2) Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0, 1]$ such that $\lim_{n\to\infty} \int_0^1 |f_n| dm = 0$ and there is an integrable function g on $[0, 1]$ such that $|f_n|^2 \leq g$, for each n.

a) Prove that $\lim_{n\to\infty} \int_0^1 |f_n|^2 dm = 0$

b) Prove that if $\lim_n f_n = f$ exists a.e. then f integrable on $[0, 1]$ and $\int f dm = 0$

3) If f is a complex valued measurable function on $(X, \mathcal{M}, \mu)$, define

$$R_f = \{z : \mu(\{x : |f(x) - z| < \epsilon\}) > 0\ \forall \epsilon > 0\}$$

Show that

a) R_f is closed. b) If $f \in L^\infty$ then R_f is compact.

4) Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space and define ν on $\mathcal{M}$ by $\nu(A) = 0$ if $\mu(A) = 0$; and $\nu(A) = \infty$ if $\mu(A) > 0$.

a) Show that ν is a measure on X and $\nu << \mu$. b) Find $\dfrac{d\nu}{d\mu}$.

GRADUATE PRELIMINARY EXAMINATION
ANALYSIS I (REAL ANALYSIS)
Fall 2005
September 12^{th}, 2005

Duration: 3 hours

1. Let (X, S, μ) be a measure space, T be a metric space. Let $f : X \times T \to \mathbf{R}$ be a function. Assume that $f(\cdot, t)$ is measurable for each $t \in T$ and $f(x, \cdot)$ is continuous for each $x \in X$. Prove that if there exists an integrable function g such that for each $t \in T$, $|f(x,t)| \leq g(x)$ for $a.a.x$, then $F : T \to \mathbf{R}$, $F(t) = \int f(x,t)d\mu(x)$ is continuous.

2. Let $\mathcal{G}$ be a set of half-open intervals in $\mathbf{R}$. Prove that $\cup_{G \in \mathcal{G}} G$ is Lebesgue measurable.

3. **a)** Let $f_n = \sin n^2 x \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $f_n \to 0$ weakly, but $f_n \not\to 0$ in measure.

b) Let $g_n = n^2 \chi_{[0, \frac{1}{n}]} \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $g_n \to 0$ in measure, but $g_n \not\to 0$ weakly.

c) Let A_n be a measurable subset of $[0,1]$ for each $n, \chi_{A_n} \in L_1$, and $\chi_{A_n} \to f$ weakly in L_1. Show that f is not necessarily a characteristic function of some measurable set.

4. Let $f : \mathbf{R} \to \mathbf{R}$. If $f \in L_1(m) \cap L_2(m)$ where m denotes the Lebesgue measure, prove that

a) $f \in L_p(m) \ \ \forall \ \ 1 \leq p \leq 2$

b) $\lim\limits_{p \to 1^+} ||f||_p = ||f||_1$.

TMS

Spring 2010

Real Analysis

1. a) Show that $f(x) = \dfrac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and $\int f d\mu = 1$

   b) A set E in $\mathbb{R}$ is said to be **null** if for any $\epsilon > 0$ we can cover E with countably many open intervals the sum of whose lengths is less than ϵ, i.e., $E \subset \cup_{n=1}^{\infty}(a_n, b_n)$ and $\sum_{1}^{\infty}(b_n - a_n) < \epsilon$.

   Show that any countable set in $\mathbb{R}$ is **null**.

2. Using Lebesgue Dominated Covergence Theorem, compute

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-kn^2}$$

   Hint: Consider $\mathbb{N}$ with the counting measure. Let

$$f_k : \mathbb{N} \to [0, \infty) \quad \text{be defined as} \quad f_k(n) = e^{-kn^2}. \quad \text{Use LDCT.}$$

3. a) Suppose $(f_n) \to f$ in measure and $(g_n) \to g$ in measure. Show $(f_n + g_n) \to f + g$ in measure.

   b) Let (f_n), (g_n) be sequences of measurable functions such that $(f_n) \to f$ in measure, $(g_n) \to g$ in measure and $f_n = g_n$ a.e. for every n. Show that $f = g$ a.e.

4. State Egoroff's theorem. Prove that in Egoroff's theorem the hypothesis $\mu(X) < \infty$ can be replaced by $|f_n| \leq g$ for all n where $g \in L^1(\mu)$

TMS

September 2011

Real Analysis

I. a) Let A_n be a sequence of measurable sets with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(\overline{lim} A_n) = 0$

Hint: $\overline{lim} A_n = \bigcap_{k=1}^{\infty} \cup_{n \geq k} A_n$

b) Let $f \in L_p(\mu)$ and $\epsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int |f|^p d\mu$$

II. a) Show that $f(x) = \frac{1}{\sqrt{x}}$ is Lebebsgue integrable over $[0, 1]$.

b) Compute $\lim_n \int_0^1 \frac{n \sin x}{1+n^2\sqrt{x}} dx$ and justify your calculations.

III. Assume $\mu(X) < \infty$. If f_n is a sequence of measurable functions on X such that $f_n \to f$ a.e. then prove that $f_n \to f$ [meas] also holds.

State the theorem(s) you used.

IV. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(x) \leq g(x)$ holds for all $x \in [a, b]$. Set $A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b]$ and $f(x) \leq y \leq g(x)\}$.

a) Show that A is a closed set (and hence a measurable subset of $\mathbb{R}^2$)

b) If $h : A \to \mathbb{R}$ is a continuous function, then show that h is Lebesque integrable over A and that

$$\int_A h d\lambda = \int_a^b (\int_{f(x)}^{g(x)} h(x, y) dy) dx$$

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2012 - TMS EXAM

1. Prove disprove:

a) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable then the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\ dm(x)$ is convergent.

b) If $\int_{-\infty}^{\infty} f(x)\ dm(x)$ is convergent then $f \in L^1$.

2. Compute $\lim_{n\to\infty} \displaystyle\sum_{k=0}^{\infty} \left(\frac{n}{2n+k}\right)^k$

(Hint: Use a convergence theorem)

3. Let $E \subset [0,1] \times [0,1]$ have the property that every horizontal section Ey is countable and every vertical section Ex has countable complement $[0,1] \setminus E_x$. Prove that E is not L-measurable.

4. Let (X, σ, μ) be a measure space.

a) Define convergence in measure

b) Let $\phi : \mathbb{C} \to \mathbb{C}$ be uniformly continuous. Let $f_n, f : X \to C$ be measurable and $f_n \to f$ in measure.

Show that $\phi \circ f_n$ converges to $\phi \circ f$ in measure.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2013 - TMS EXAM

1. (35 pts.) Denote by χ_A the characteristic function of $A \subseteq [0,1]$

a) Prove that $\psi(t,x) := (t, \frac{x+\chi_A(t)}{2})$ is measurable if and only if A is measurable

b) Suppose A is measurable, calculate the integral $\int_{[0,1]\times[0,1]} \psi d\mu$ where μ is the Lebesgue measure on $[0,1]\times[0,1]$

c) Give an example of $A \subseteq [0,1]$ which is not Lebesgue measurable.

2. (20 pts.) Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma, the monotone and the dominated convergence theorems as statements about infinite series.

3. (25 pts.) a) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which maps a Lebesgue measurable set onto a non-Lebesgue measurable set.

b) Why the condinition $|f_n| \leq g \in L_1$ in the Dominated convergence theorem cannot be replaced by $|f_n(t)| \leq M \in \mathbb{R}^+$.

4. (20 pts.) Given the counting measure ν on $P(\mathbb{R})$ and the Lebesgue measure μ on the Lebesgue algebra $\sum(\mathbb{R})$.

a) Show that μ is absolutely continuous with respect to ν.

b) Explain why the Radon-Nikodym theorem is not applicable to measures ν and μ.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2014 - TMS EXAM

1.

(a) State the Lebesgue Dominated Convergence Theorem.

(b) Use (a) to evaluate

$$\lim_{n\to\infty}\int_0^1 \frac{dx}{\cos(x+\frac{1}{n})\,x^{\frac{1}{n}}}$$

where dx denotes integration with respect to Lebesgue measure.

[Be sure to explain why the hypotheses are satisfied when you quote (a).]

2. Either prove or provide an explicit counterexample to each of the following assertions: (you may quote without proof familiar relations and containments between L^p-spaces)

(a) If $f, g \in L^2([0,1])$ then $fg \in L^2([0,1])$. (Lebesgue measure)

(b) If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$. (Lebesgue measure)

(c) If $f, g \in L^2(\mathbb{R})$ then $fg \in \ell^2$. (counting measure)

3. Let λ denote Lebesgue measure on the real line.

(a) Prove that there is an open set $\mathcal{O}$ that is dense in $\mathbb{R}$ with $\lambda(\mathcal{O}) < 1$.

(b) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is uncountable.

(c) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is not compact.

4. Let m be Lebesgue measure on $[0,1]$ and n be counting measure and $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

(a) Show $\int\int f(x,y)dm(x)dn(y) \neq \int\int f(x,y)dn(y)dm(x)$.

(b) State the Fubini-Tonelli Theorem and state why the above result does not contradict the Theorem.

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2015 - TMS EXAM

1. Formulate the Egoroff theorem (= the third Littlewood principle) and show that it fails in every measure space with infinite σ-finite measure.
Hint: Consider $f_n = \chi_{[n,n+1]}$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and suppose $X = \bigcup_n X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of measurable subsets of X. Use the monotone convergence theorem and the linearity of the integral to prove that, if f is a non-negative measurable real-valued function on X,

$$\int_X f d\mu = \sum_n \int_{X_n} f d\mu$$

Hint: Let $f_n = \displaystyle\sum_{k=1}^{n} f\ \chi_{X_k} = f\chi_{\cup_1^n X_k}$

3. Evaluate $\displaystyle\lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-kn^2}$ and prove your answer by using a measure theory theorem.

Hint: Let $f_k : \mathbb{N} \to [0, \infty)$ be defined by $f_k(n) = e^{-kn^2}, n \in \mathbb{N}$.

4. Using the Fubini/Tonelli theorems to justify all steps, evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos(\frac{\pi y}{2x}) dx dy$$

Hint: Consider $\int\int \ |x^{-3/2} \cos(\frac{\pi y}{2x})| dy dx$ and apply to Tonelli's theorem.