Graduate Preliminary Examination Topology Duration: 3 hours

1. Consider a topological space X, Y and a continuous map $f: X \to Y$.

a) Prove that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset B of Y.

b) Suppose that f is also closed and surjective. Prove that $\overline{B} = f(\overline{f^{-1}(B)})$ for any subset B of Y.

c) Suppose that X is metrisable and f is a closed, surjective and continuous map. Prove that for any subset B of Y and any $y \in \overline{B}$ there exists a sequence $y_n \in B$ such that $\lim y_n = y$.

2.

a) Is the intersection of two dense subsets in a topological space always dense?

b) Let X be a topological space. Prove that the intersection of two open dense subsets of X is open and dense.

c) If H
 is the family of open dense subsets in X, prove that H = H
 ∪ {φ} is a topology on X.

d) Let \hat{X} be the topological space which consists of the set X with the topology \mathcal{H} on it. Prove that a function $f: \hat{X} \to \mathbb{R}$ is continuous iff it is constant.

 Let f be a continuous mapping of the compact space X onto the Hausdorff space Y. Show that any mapping g of Y into Z for which g ◦ f is continuous must itself be continuous.

4. Consider the cyclinder $S^1 \times I$ where S^1 the unit circle in \mathbb{R}^2 and I = [0, 1]. Identify $S^1 \times \{1\}$ to a point i.e. define an equivalence relation

~ on S^1 by letting $(u, 1) \sim (v, 1)$ for all $u, v \in S^1$ and letting all other elements in $S^1 \times [0, 1]$ be related only to itself. Show that the quotient space. $(S^1 \times I)/\sim$, the so called cone on S^1 , is homomorphic to the unit disc D^2 in \mathbb{R}^2 .

METU-MATHEMATICS DEPARTMENT Graduate Preliminary Examinations

Topology

Duration: 3 hours

February 21, 2005

- 1. A collection $\{f_{\alpha} \mid \alpha \in A\}$ of functions on a space X (to spaces X_{α}) is said to separate points from closed sets in X iff whenever B is closed in X and $x \notin B$, then for some $\alpha \in A$, $f_{\alpha}(x) \notin \overline{f_{\alpha}(B)}$. Then, prove that a collection $\{f_{\alpha} \mid \alpha \in A\}$ of continuous functions on a topological space X separates points from closed sets in X if and only if the sets $f_{\alpha}^{-1}(V)$, for $\alpha \in A$ and V open in X_{α} , form a base for the topology on X.
- **2.** Let X be a compact space and let $\{C_{\alpha} \mid \alpha \in A\}$ be a collection of closed sets, closed with respect to finite intersections. Let $C = \cap C_{\alpha}$ and suppose that $C \subset U$ with U open. Show that $C_{\alpha} \subset U$ for some $\alpha \in A$.
- **3.** Let (X, d) be a metric space.

(a) Consider a connected set $A \subset X$ and a continuous function $f : X \to \mathbb{R}$. Given $\alpha, \beta \in f(A) \subset \mathbb{R}$ with $\alpha \leq \beta$ prove that for every $t \in \mathbb{R}$ with $\alpha \leq t \leq \beta$ there exists $a \in A$ such that f(a) = t.

(b) For any $B \subset X$, prove that the function $g: X \to \mathbb{R}$ defined by

$$g(x) = \operatorname{dist}(x, B) = \inf\{d(x, b) \mid b \in B\}$$

for all $x \in X$ is continuous.

(c) Let $\Omega \subset X$ be open, connected and relatively compact (i.e. its closure is compact). Consider a continuous surjective function $h : \Omega \to \Omega$. Prove that there exists $w \in \Omega$ such that

$$\operatorname{dist}(w, \operatorname{Bd}(\Omega)) = \operatorname{dist}(h(w), \operatorname{Bd}(\Omega))$$

(Hint : Consider the point $w\in\overline{\Omega}$ at which ${\rm dist}(w,{\rm Bd}(\Omega))$ achieves its maximum.)

4. Let $X = \mathbb{R} \cup \{\infty\}$ with the topology with respect to which a subset of X not containing ∞ is open if it is open in the usual sense, a subset

of X containing ∞ is open if its complement is the union of finitely many sequences convergent in the usual sense along with their limits.

(a) Prove that for any open set U in X the set $U \setminus \{\infty\}$ is open in \mathbb{R} (Hint: the terms of a convergent sequence in \mathbb{R} together with its limit is a closed set).

- (b) Prove that $[0,1] \subset X$ is compact.
- (c) Prove that $[0,1] \subset X$ is not closed. Is X Hausdorff?

(d)Prove that for every continuous $f : X \to X$ with $f(\infty) = \infty$, the set of fixed points $\{x \in X \mid f(x) = x\}$ is closed.

METU - Mathematics Department Graduate Preliminary Exam

Topology

Duration : 3 hr.

Feb 9, 2007

- 1. Let τ be a Hausdorff topology on a set $X \neq \emptyset$ and for $A \subset X$, let \overline{A} denote the closure of A with respect to τ .
 - a) Prove that the family

 $\mathcal{B} = \{ U - A : U \text{ is open, } \overline{A} \text{ is compact in } \tau \}$

constitutes a basis for a new topology τ *.

b) Prove that $\tau *$ is Hausdorff.

c) Prove that $\tau = \tau *$ if and only all subsets of X which are compact in τ , are finite subsets.

2. Let X be an infinite set with the finite complement topology (i.e. the collection of open sets is $\tau = \{A : X - A \text{ is finite, or } A = \emptyset\}$).

a) Prove that every subset of X is compact.

b) Prove that X is T_1 (i.e. For every $x, y \in X$ with $x \neq y$, there are open sets U, V such that $x \in U - V$ and $y \in V - U$).

Is X Hausdorff? Is X metrizable?

c) If $X = \mathbb{R}$, find the closures and interiors of $(0, 1], [2, 3], \mathbb{Z}$.

- 3. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X. Consider the following collection Ω^* of subsets of X^*
 - (i) open sets in X
 - (ii) sets of the form $X^* S$ where S is a compact subset of X.

Prove the following statements for the topological space (X^*, Ω^*) (do not prove that Ω^* defines a topology on X^*).

a) (X^*, Ω^*) is compact.

b) If X is locally compact, then X^* is Hausdorff.

c) A continuous map $f: X \to Y$ between Hausdorff topological spaces extends to a map $f^*: X^* \to Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

d) If Y is a locally compact Hausdorff space and if $f: X \to Y$ is proper, then f is a closed map.

4. Let (X, d) be a metric space. For a point x and for subspaces A, B in X, define $d(x, A) = \inf\{d(x, a) : a \in A\}$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

(a) Prove that the map $f: X \to \mathbb{R}$ defined by f(x) = d(x, A) is continuous.

(b) Prove that if A is compact, then there is a point $a_0 \in A$ such that $d(x, a_0) = d(x, A)$.

(c) Prove that if A and B are compact, then there are points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = d(A, B)$.

(d) Prove that A and B are compact and disjoint, then there are disjoint open sets U and V in X such that U contains A and V contains B.

(e) Show that the conclusion of (b) may not be true if A is not compact.

METU - Mathematics Department Graduate Preliminary Exam-Spring 2008

Topology

1. Consider the real line $\ensuremath{\mathbb{R}}$ with the usual topology. Let $\ \sim$ be the equivalence relation defined by

 $x \sim y$ if and only if $x - y \in \mathbb{Q}$.

Show that the quotient space \mathbb{R}/\sim has uncountable number of elements and that its topology is trivial.

2. A discrete valued map on a topological space X is a continuous map $X \to D$ into a discrete topological space D. Show that

a) X is connected if and only if every discrete valued map on X is constant.

b) the statement "d(p) =d(q) for every discrete valued map d on X" defines an equivalence relation on X and that the corresponding equivalence classes are closed subsets of X.

3. Let X and Y be two topological spaces and let $X \times Y$ be given the product topology.

a) Suppose K is a compact subset of X and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that y has a neighborhood $U \subset Y$ such that $K \times U \subset A$.

b) (i) Suppose X is compact. Prove that the projection $\pi: X \times Y \to Y$ is a closed map.

(ii) Give an example to show that in (i) the *compactness* assumption is essential.

- 4. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X. Consider the following collection Ω^* of subsets of X^*
 - (i) open sets in X
 - (ii) sets of the form $X^* S$ where S is a compact subset of X.

Prove the following statements for the topological space (X^*, Ω^*) (do not prove that Ω^* defines a topology on X^*).

a) (X^*, Ω^*) is compact.

b) If X is locally compact, then X^* is Hausdorff.

c) A continuous map $f: X \to Y$ between Hausdorff topological spaces extends to a map $f^*: X^* \to Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

d) If X, and Y are locally compact Hausdorff spaces and if $f : X \to Y$ is proper and continuous, then f is a closed map.

GRADUATE PRELIMINARY EXAMINATION GENERAL TOPOLOGY Duration: 3 hours February 10th, 2009

- 1. A topological space X is said to be hyperconnected if every nonempty open set in X is dense. A topological space Y is said to be ultraconnected if $\overline{\{a\}} \cap \overline{\{b\}} \neq \emptyset$ for every $a, b \in Y$.
 - a) Prove that a hyperconnected topological space is connected.

b) Prove that an infinite set with the cofinite topology is hyperconnected but not ultraconnected.

c) Let Z be a topological space with more than two points and $p \in Z$ where $U \subseteq Z$ is open iff U = Z or $p \notin U$. Prove that Z is ultraconnected but not hyperconnected.

d) Prove that an ultraconnected space Y is path connected by demonstrating that for any $a, b \in Y$ and any $p \in \overline{\{a\}} \cap \overline{\{b\}}$ the map $\lambda : [0, 1] \to Y$ defined by

$$\lambda(t) = \begin{cases} a & \text{if } 0 \le t < 1/2 \\ p & \text{if } t = 1/2 \\ b & \text{if } 1/2 < t \le 1 \end{cases}$$

is continuous.

2. A continuous map $f: X \to Y$ is called a (topological) embedding if the map $\tilde{f}: X \to f(X)$ obtained by restricting the range of f is a homeomorphism (Here f(X) has the subspace topology).

a) Show that \mathbb{Q} cannot be embedded in \mathbb{Z} (where both has the subspace topology of \mathbb{R})

b) Let \mathbb{R}_c denote the topological space whose underlying set is \mathbb{R} and its open sets are complements of finite sets and the empty set.

(i) If $Y \subset \mathbb{R}_c$, describe open subsets of Y.

(ii) Show that \mathbb{R} with its usual topology cannot be embedded in \mathbb{R}_c .

- 3. a) Let X and Y be metric spaces so that Y is compact. Show that the projection map π₁: X × Y → X, where π₁(x, y) = x for all (x, y) ∈ X × Y, is a closed map.
 b) Let X = Y = ℝ equipped with the usual metric. Show that the projection map π₁: ℝ × ℝ → ℝ in Part (a) is not a closed map.
- 4. Let $A = \{A_i | i \in I\}$ be a cover for topological space X such that each $p \in X$ has a neighborhood N such that $\{i | N \cap A_i \neq \phi\}$ is finite and $f : X \to Y$ be a function where Y is a topological space.

a) Show that $\overline{\bigcup_{i\in I} B_i} = \bigcup_{i\in I} \overline{B_i}$ where $B_i \subseteq A_i$ for each $i \in I$.

b) Show that f is continuous when $f|A_i$ continuous and A_i is closed for each $i \in I$.

c) Show that $\{i|K \cap A_i \neq \phi\}$ is finite for any compact subset K of X.

d) Show that the component of a point $p \in X$ (maximal connected subset which contains p) is open when A_i is connected for each $i \in I$.

\mathbf{TMS}

Spring 2010

TOPOLOGY

- I. Show that for any map f from a topological space E into a topological space F, the following are equivalent:
 - a) f is continuous on E.
 - b) f^{-1} (Int A) \subset Int $f^{-1}(A)$ holds for every set $A \subset F$.
 - c) $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$ holds for every set $A \subset F$.
- II. Let E be the set of all ordered pairs (m, n) of non-negative integers. Topologize E as follows:

For a point $(m, n) \neq (0, 0)$ any set containing (m, n) is a neighborhood of (m, n).

A set U containing (0,0) is a neighborhood of (0,0) if and only if for all except a finite number of m's the set $\{n: (m,n) \notin U\}$ is finite.

- a) Show that E is not locally compact.
- b) Show that E is normal.
- III. Let X be a topological space and A, B be subsets of X. Show that:
 - a) If $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi$ and if C is a connected set which is contained in $A \cup B$ then either $C \subset A$ or $C \subset B$.
 - b) If $(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \phi$ and if A and B are connected then $A \cup B$ is also connected.
- IV. Let f be a continuous one-to-one map from a compact space E into a Hausdorff space F.
 - a) Show that $f^{-1}: f(E) \to (E)$ is continuous
 - b) If $A \subset f(E)$, then prove $\overline{A} \subset f(\overline{f^{-1}(A)})$.

c) Let (x_n) be a sequence of real numbers. If $\lim(2x_n + \sin x_n) = \frac{\pi}{3} + \frac{1}{2}$ then prove that $\lim x_n = \pi/6$

Topology, February 2012 TMS EXAM February 15, 2012

1. Let $\mathcal{T} = \{ U \cup -U \mid U \subset \mathbb{R} \text{ is an open set in standard topology of } \mathbb{R} \}$. Show that \mathcal{T} is a topology on \mathbb{R} and find the interiors and the closures of the following intervals of \mathbb{R} in this topology: $A = (-2, 3), B = (-\infty, 3), C = (1, 3).$

2. Let X be a topological space. Prove the given statement or give a counter example to show that it is not always true.

- a) If $A \subset X$ is connected then \overline{A} is also connected.
- b) If $A \subset X$ is path connected then \overline{A} is also path connected.
- 3. Consider the space $X = \{a, b, c\}$ with the topology $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
 - a) Show that $Y = \{1, 2\}$ with topology $\{\emptyset, Y, \{1\}, \{2\}\}$ is not a quotient space of X.
 - b) Show that $Y = \{1, 2\}$ with topology $\{\emptyset, Y, \{1\}\}$ is a quotient space of X.
 - c) Consider the function $f: X \to Y$, f(a) = 1, f(b) = 1 and f(c) = 2, where Y has the indiscrete topology $\{\emptyset, Y\}$. Is f a quotient function? Is Y is a quotient space of X?

4. Let X be a countable set (you may take, for example, $X = \mathbb{Z}_+$) equipped with the discrete topology. Find a homeomorphism $f: X^+ \to A$, where A is the subspace of \mathbb{R} given by

$$A = \{\frac{1}{n} \mid n = 1, 2, \dots\} \cup \{0\}$$

and X^+ is the one point compactification of X.

Topology February 2013 Middle East Technical University

Preliminary Exam DURATION: 3 hours

1- Prove or disprove. $(X, Y \text{ are topological spaces}, A, B \text{ are subsets of a topological space } X, \overline{A}$ denotes the closure of the set A, A' denotes the set of limit points of the set A.)

- (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- (b) f(A') = f(A)' for any continuous function $f: X \to Y$.
- (c) If X and Y are Hausdorff, then $X \times Y$ is Hausdorff in the product topology.
- (d) If X is a metric space then X is Hausdorff.
- (e) A infinite set X with the finite complement topology (that means a set $U \subset X$ is open if X U is finite) is metrizable.

2- For a point $x = (x_1, x_2) \in \mathbb{R}^2$ and a positive number r > 0, define

$$B(x,r) = \{ y = (y_1, y_2) \in \mathbb{R}^2 \mid |x_1 - y_1| < r \}.$$

Let $\mathcal{B} = \{ B(x, r) | x \in \mathbb{R}^2, r > 0 \}.$

- (a) Show that \mathcal{B} is a basis for a topology on \mathbb{R}^2
- (b) Compare this topology with the standard topology on \mathbb{R}^2 .
- (c) Let $D = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$. Find the interior D° and the closure \overline{D} of the set D in this topology.

3- Let $\pi: X \times Y \to X$ be the projection map onto the first coordinate. Show that if Y is compact then π is a closed map.

4- Let X, Y be metric spaces. Using the definition of continuity ONLY, show that a function $f: X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Topology TMS EXAM February 18, 2014

Duration: 3 hours

1. A topological space is *extremally disconnected* if and only if the closure of every open set is open. Show that for any topological space X the following are equivalent:

- (a) X is extremally disconnected,
- (b) Every two disjoint open sets in X have disjoint closures.

2. Show the following:

- (a) An open subset of a separable space is separable.
- (b) The product of countable number of separable spaces is separable.
- (c) The quotient space of a separable space is separable.

3. Let X, Y be topological spaces and $f : X \to Y$ be a continuous map. Consider the graph $G = \{(x, f(x)) : x \in X\}$ of f with the subspace topology of $X \times Y$.

- (a) Show that G is homeomorphic to X.
- (b) Show that G is closed if Y is Hausdorff.

4. Let X be a compact Hausdorff space and $f: X \to Y$ be a quotient map. Show that the following are equivalent:

- (a) Y is an Hausdorff space,
- (b) f is a closed map,
- (c) The set $\{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$.

TOPOLOGY TMS EXAM February 11 2015

Duration: 3 hours.

(1) Consider the set of real numbers \mathbb{R} . The collection of all open intervals (a, b) along with all sets of the form (a, b) - K where $K = \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is a basis for a topology on \mathbb{R} called the K-topology. Let \mathbb{R}_K denote \mathbb{R} with this topology.

(a) Compare \mathbb{R}_K with the standard topology on \mathbb{R} .

(b) Is \mathbb{R}_K compact? Is [0,1] a compact subspace of \mathbb{R}_K ?

(c) Is \mathbb{R}_K Hausdorff?

(d) Is \mathbb{R}_K regular?

(e) Is the quotient space of \mathbb{R}_K obtained by collapsing the set K to a point Hausdorff? Is it T_1 ?

(2) Let X be a compact metric space and suppose that $f: X \to X$ is an isometry, that is d(f(x), f(y)) = d(x, y) for all $x, y \in X$. Prove that f is a homeomorphism.

(3) Let \mathbb{R}^{ω} be the countably infinite product of \mathbb{R} with itself and let $A \subset \mathbb{R}^{\omega}$ be defined by

$$A = \{ (x_i) \in \mathbb{R}^{\omega} | x_i = 0 \text{ for all but finitely many } i \}.$$

(a) Prove that A is dense in \mathbb{R}^{ω} with the product topology.

(b) Prove that A is not dense in \mathbb{R}^{ω} with the box topology.

(4) Let $r: S^1 \to S^1$ be defined by r(x, y) = (-x, y). The Klein bottle K is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, (x, y)) \sim (1, r(x, y))$ for all $(x, y) \in S^1$ and (t, (x, y)) is not equal to anything except itself for $t \neq 0, 1$.



(a) Show that K is compact.

(b) Let $C_1 \subset K$ be (the image of) the circle $\{\frac{1}{3}\} \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside (b) Let $C_1 \subseteq R$ be (the image of) the ended $(3) \neq 0$, and r = 1 be (the image of) the ended $(3) \neq 0$, $(1, 3) \neq 0$. $(\frac{1}{2}, \frac{3}{4}) \times S^1$. There is a continuous map $g: K \to \mathbb{R}^3$ as shown in the picture. The restriction of g to $K - C_1$ is injective so is the restriction to $K - C_2$, but $g(C_1) = g(C_2)$. Assuming that such a map gexists as described, use Urysohn's Lemma to construct a continuous map of K into $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ which is an embedding. You may assume that K is Hausdorff.



METU-MATHEMATICS DEPARTMENT Graduate Preliminary Examinations

Topology

Duration: 3 hours

September 24, 2004

- **1.** Let τ be a Hausdorff topology on X. Let $Cl_{\tau}(A)$ denote the closure of $A \subseteq X$ with respect to the topology τ
 - (a) Prove that the family

 $\mathfrak{B} = \{ U - A \mid U \text{ is open, } Cl_{\tau}(A) \text{ is compact in } \tau \}$

constitutes a basis for a new topology τ^* on X.

- (b) Prove that τ^* is Hausdorff.
- (c) Prove that $\tau = \tau^*$ iff all compact subsets of τ are finite.
- **2.** If $A \times B$ is a compact subset of $X \times Y$ contained in an open set W in $X \times Y$, then there exists open sets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq W$.
- **3.** Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of Hausdorff infinite spaces, and $X = \prod_{i \in \mathbb{N}} X_i$ with the topology which has a base $\{\prod_{i \in \mathbb{N}} U_i : U_i \text{ open}\}$. Show that
 - (a) X is Hausdorff
 - (**b**) X is not seperable
 - (c) X has a closed discrete subspace which has the cardinality of \mathbb{R} .
- **4.** Let $X = (\mathbb{R}^2, T)$ where T is the topology on \mathbb{R}^2 which is induced by the following metric

$$d((p,q),(r,s)) = \begin{cases} |q-s| & \text{if } p = r \\ |q| + |p-r| + |s| & \text{if } p \neq r \end{cases}$$

- (a) Describe the relative topologies on $\mathbb{R} \times \{0\}$, $\{p\} \times \mathbb{R}$, $\mathbb{R} \times \{q\}$ for $q \neq 0$, and on an arbitrary line ax + by = c.
- (b) Show that X is path connected.
- (c) Find the number of path components of $X \setminus \{(p,q)\}$

METU - Mathematics Department Graduate Preliminary Exam

Topology

Duration : 3 hours

Fall 2005

- 1. Let (X, d_x) , (Y, d_Y) be two metric spaces and let $f : X \to Y$ be continuous in the usual ϵ , δ definition.
 - a) Prove that for every open set $V \subset Y$, the set $f^{-1}(V)$ is open in X.

b) Suppose that X is compact and $y_0 \in Y - f(X)$. Prove that there is an open neighborhood V of f(X) and a positive number r such that $V \cap B(y_0; r) = \emptyset$. Here, $B(y_0; r) = \{y \in Y : d_Y(y, y_0) < r\}$.

- 2. Let X be an infinite set with the finite complement topology (i.e. the collection of open sets is $\tau = \{A : X A \text{ is finite, or } A = \emptyset\}$).
 - **a)** Prove that every subset of X is compact.

b) Prove that X is T_1 (ie. For every $x, y \in X$ with $x \neq y$, there are open sets U, V such that $x \in U - V$ and $y \in V - U$).

- Is X Hausdorff ? Is X metrizable ?
- c) If $X = \mathbb{R}$, find the closures and interiors of $(0, 1], [2, 3], \mathbb{Z}$.
- 3. a) Consider the following subsets of \mathbb{R}^2 :

$$X = \{(\pm \frac{1}{n}, y) : n \ge 1, 0 \le y \le 1\} \cup \{(x, 0) : |x| \le 1\} \cup \{(0, y) : 0 \le y \le 1\},\$$
$$Y = \{(\pm \frac{1}{n}, y) : n \ge 1, 0 \le y \le 1\} \cup \{(x, 0) : |x| \le 1\} \cup \{(-2, y) : 0 \le y \le 1\}$$

Show that in the topology induced from \mathbb{R}^2 ,

- (i) X is connected but not locally connected, and
- (ii) Y is locally connected.

b) Show that the image of a locally connected set under a continuous map is not necessarily locally connected.

Hint : Consider the map $f: Y \to X$, $f(a, b) = \begin{cases} (a, b) & \text{if } a \neq -2 \\ (0, b) & \text{if } a = -2. \end{cases}$

c) Show that a compact Hausdorff space is locally connected if and only if every open cover of it can be refined by a cover consisting of a finite number of connected spaces.

4. a) Show that a topological space X is regular if and only if for each $x \in X$ and any neighborhood U of x, there is a closed neighborhood V of x such that $V \subset U$.

b) Let X be a regular space and let \mathcal{D} be the family of all subsets of the form $\overline{\{x\}}$ where $\overline{\{x\}}$ denotes the closure of the point $x \in X$. Show that \mathcal{D} is a partition of X.

c) Show that, in the quotient topology induced by the projection

 $p: X \to \mathcal{D}, \ p(x) = \overline{\{x\}}, \ \mathcal{D}$ is a regular Hausdorff space.

METU - Mathematics Department Graduate Preliminary Exam

Topology

Duration : 3 hours

September 15, 2006

- 1. Let X be an infinite set and $T = \{A \subset X : A = \emptyset \text{ or } X A \text{ is finite}\}$. Let Y be a **subspace** of the topological space (X, T). Show that
 - a) Y is T_2 if and only if Y is finite.
 - b) Y is connected if and only if Y is infinite.
 - c) Y is compact and separable.

d) If the cardinality of Y is at least the cardinality of \mathbb{R} , then Y is path connected.

- 2. Let X, Y be topological spaces and $f : X \to Y$ be a continuous, open surjection and \mathcal{B} be a base for the topological space X. Show that
 - a) The set $\{f(U) : U \in \mathcal{B}\}$ is a base for the topological space Y.

b) If X is locally connected, then Y is locally connected.

c) If X is locally compact (not necessarily Hausdorff), then Y is locally compact.

3. Let $X \neq \emptyset$ be a topological space. We say $X_0 \subset X$ is very dense in X if the following correspondence between the sets of open subsets

$$\mathcal{O}(X) \to \mathcal{O}(X_0)$$
$$U \mapsto U \cap X_0$$

is injective.

a) (i) Show that a v.d. set is dense and give a **non-Hausdorff** example to show that the converse is not true.

(ii) Determine the topology of X if $\{x\} \subset X$ is v.d.

b) Show that if X is T_0 and contains a v.d. subset X_0 which is minimal (with respect to inclusion) in the set of nonempty closed subsets of X, then X consists of a single point.

c) True or false ? Explain (prove the claim or give a counter example).

If X is a connected topological space and if X_0 is v.d. in X, then X_0 is connected too.

4. Let C(X, Y) be the set of all continuous functions between a locally compact topological space X and an arbitrary topological space Y. We define the **weak topology** on C(X, Y) by taking as **subbase** the sets of the form

$$\{f \in C(X, Y) : f(K) \subset V\}$$

for compact subsets $K \subset X$ and open sets $V \subset Y$.

A) Suppose X is compact and Y is a metric space. Show that

- a) the weak topology is defined by a metric d_W on C(X, Y),
- b) if Y is complete, then $(C(X, Y), d_W)$ is a complete metric space.

B) Show that if $g: X_1 \to X_2$ is a continuous map between locally compact spaces then the induced map

$$g^*: C(X_2, Y) \to C(X_1, Y)$$
$$f \mapsto f \circ g$$

is continuous.

METU - Mathematics Department Graduate Preliminary Exam-Fall 2007

Topology

1. Let X and Y be two topological spaces and let $X \times Y$ be given the product topology.

a) Suppose K is a compact subset of X and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that y has a neighborhood $U \subset Y$ such that $K \times U \subset A$.

b) (i) Suppose X is compact. Prove that the projection $\pi : X \times Y \to Y$ is a closed map.

(ii) Give an example to show that in (i) the *compactness* assumption is essential.

2. a) Let X be a connected, Hausdorff and completely regular topological space. Prove that if X has at least two points, then X is uncountable.

(Recall : A T_1 -topological space X is completely regular if for any given closed subset Y of X and for any $p \in X$, one can find a continuous function $f: X \to \mathbb{R}$ such that f(p) = 0 and $f|_Y = 1$.)

b) Give an example of a Hausdorff completely regular space which is countable.

- 3. Prove that the one-point compactification of \mathbb{R}^2 (in its usual topology) is homeomorphic to the 2-sphere $S^2 \subset \mathbb{R}^3$ (the topology on S^2 is induced from \mathbb{R}^3).
- 4. True or false ? Prove the statement or give a counter example.

a) Let (X, d) be a metric space. Suppose that there exist a point $a \in X$ and a real number $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$ one has $B(a; \epsilon) = \overline{B(a; \epsilon)}$. Then (X, d) is bounded.

b) If $g: Y \to X$ is a continuous map into a discrete topological space, then g is constant on each connected component of Y.

c) Recall that a point $x \in X$ in a topological space X is called a *generic point* if it is dense.

(i) If X has a unique generic point, then it is connected.

(ii) A connected topological may have at most a unique generic point.

METU - Department of Mathematics Graduate Preliminary Exam General Topology

Fall 2008

1.

(A) In a topological space, is the intersection of two dense subsets always dense ?

(B) Let X be a topological space. Prove that the intersection of two open and dense subsets of X is open and dense.

(C) Let \mathfrak{H} be the family subsets of X consisting of the open and dense subsets of X and the empty set. Prove that \mathfrak{H} is a topology on X.

(D) Let \widetilde{X} be the topological space with carrier set X and topology \mathfrak{H} . Let Y be a topological space and \widetilde{Y} be defined similarly. If $f: X \longrightarrow Y$ is a continuous and open map, prove that $f: \widetilde{X} \longrightarrow \widetilde{Y}$ is continuous.

2.

(A) Prove that a closed subset of a compact topological space is compact.

(B) Prove that a compact subset of a Hausdorff topological space is closed.

(C) $\mathfrak{H}, \mathfrak{H}'$ be topologies on X such that $\mathfrak{H}' \subseteq \mathfrak{H}$. If $K \subseteq X$ is compact with respect to \mathfrak{H} , prove that K is compact with respect to \mathfrak{H}' as well.

(D) Suppose that \mathfrak{T} is a topology on X such that the topological space (X,\mathfrak{T}) is compact. If \mathfrak{S} is a topology on X which is strictly coarser than \mathfrak{T} , (that is $\mathfrak{S} \subseteq \mathfrak{T}$ yet $\mathfrak{S} \neq \mathfrak{T}$) prove that the topological space (X,\mathfrak{S}) is not Hausdorff. (*Hint*: There exists $C \subseteq X$ which is closed with respect to \mathfrak{T} but not closed with respect to \mathfrak{S} .)

3. Let $f : X \to Y$ be a continuous closed surjection. Prove the following statements:

i) If X is T_1 then Y is T_1 .

ii) If X is normal then Y is normal.

iii) If Y is connected and $f^{-1}(y)$ is connected for each $y \in Y$ then X is connected.

iv) If $f^{-1}(y)$ is Lindelöf for each $y \in Y$ and Y is Lindelöf then X is Lindelöf,

4.Let $f: X \to Y$ be a closed open surjection.

a) Let $\varphi : X \to (0,1)$ be a continuous function and $\Theta : Y \to \mathbb{R}$ be a function such that $\Theta(y) = \sup[\varphi(x)|f(x) = y]$ for each $y \in Y$. Show that θ is continuous.

b) Show that Y is regular (T_3) when X is regular and f is continuous.

c) Show that Y is $T_3\frac{1}{2}$ when X is $T_3\frac{1}{2}$ and f is continuous.

GRADUATE PRELIMINARY EXAMINATION GENERAL TOPOLOGY Duration: 3 hours Fall 2009

1. a) Let A be a subspace of X and U is an open subset of X. Show that if $A \cap U$ is closed in U then $U \cap A = U \cap \overline{A}$ where $\overline{A} = C\ell_{\mathbf{x}}A$.

b) Suppose that a subspace A of a topological space has the property that each of its points has a neighborhood U such that $A \cap U$ is closed in U. Show that A is open in \overline{A} , where \overline{A} is closure of A in X as in part(a).

c) Show that if A has the property given in part(b), then A can be written as a the intersection of an open and a closed set.

- 2. Let X be a compact connected Hausdoff space and $f: X \to X$ a continuous open map. Show that f is onto.
- 3. Let X be a space which is not Lindelöf. Adjoin a point p to X to obtain a new space \tilde{X} whose the neighborhoods of p are the sets of the form $\{p\} \cup E$ where E an open subset of X whose complement is Lindelöf. Call this new space \tilde{X} .

a) Show that X is a dense subspace of \bar{X} .

- b) Show that \tilde{X} is Lindelöf.
- 4. Let $X = \{x | x : \mathbb{N} \to \mathbb{R}\}$ with the box topology (the topology which has a base $B = \{\prod_{i \in \mathbb{N}} O_i | O_i \text{ open in } \mathbb{R} \text{ for } i \in \mathbb{N}\}.$
 - a) Show that $X \times X \to X$ given by $(x, y) \to x + y$ is continuous.
 - b) Show that $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$ given by $(r, x) \to rx$ is discontinuous.
 - c) Describe the path component of O = (0).

TMS Fall 2010 Topology

- 1. Let \mathcal{B} be the collection of all open intervals (a, b) for $a \cdot b \geq 0$, and all differences (a, b) Afor $a \cdot b < 0$ where $A = \{1/n | n = 1, 2, \cdots\}$. Show that \mathcal{B} is a basis for a topology on \mathbb{R} . Determine \overline{A} in this topology.
- 2. A topological space is said to be symmetric if $x \in \overline{\{y\}}$ happens if and only if $y \in \overline{\{x\}}$, where \overline{A} denotes the closure of A in X for any subset A of X.
 - a) Give an example of a space which is symmetric but not Hausdorff.

b) Consider the topology on \mathbb{R} in which open sets are open intervals of the form (a, ∞) for $a \in \mathbb{R}$. Show that this topology is normal but not regular. (In this question the definition of normality and regularity of a space does not assume the space is T_1).

c) Prove that a symmetric normal space is regular.

3. Let A = {1,2,3} be equipped with the discrete metric d, (i.e. d(x,y) = 1 if x ≠ y,
0 otherwise), and let the Cartesian product E = [0,1] × A be equipped with the d_∞-metric i.e.

$$d_{\infty}((x,a),(x',a')) = \max\{|x-x'|,d(a,a')\}.$$

Give examples of **compact**, **non-compact**, **connected** and **disconnected** subsets of *E* and justify your claims.

4. All the spaces in this questions are subspaces of the real line equipped with the standard topology.

a) Use connectedness to show that the any two intervals of the form [a, b) and (c, d) are not homeomorphic.

b) Show that any bijection $f: [0,1) \to (0,1)$ must have infinitely many points of discontinuity.

M.E.T.U

Department of Mathematics Preliminary Exam - Sep. 2011 Topology

Duration : 3 hr.

Each question is 25 pt.

- 1. For a subset A in topological space X, let \overline{A} denote the closure and A° denote the interior of A in X. Prove or disprove the followings:
 - (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
 - (b) $(A \cup B)^{\circ} = A^{\circ} \cup B^{\circ}$.
 - (c) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
 - (d) Every quotient space of a Hausdorff space is Hausdorff.
 - (e) Every quotient map $p: X \to Y$ is open.
 - (f) An infinite set X with the finite complement topology is metrizable.
 - (g) $(-\infty, 0)$ is homeomorphic to (0, 1).
 - (h) $(-\infty, 0)$ is homeomorphic to [0, 1).
- 2. (a) Let X be a topological space and let A be a subset of X. Give the definition of the connectedness and the path-connectedness of A.

(b) Prove that a path-connected subset of a topological space is connected.

(c) Show that the converse of (b) is not true.

(d) Let X be a topological space, C be a connected subset and E be an arbitrary subset of X. Suppose that $C \cap E \neq \emptyset$ and $C \cap (X - E) \neq \emptyset$. Show that $C \cap \partial E \neq \emptyset$, where ∂E denotes the boundary of E.

(e) Let *D* denote the subset $\{(x_1, x_2, \dots, x_n, 0) : x_1^2 + x_2^2 + \dots + x_n^2 \neq 1\}$ in \mathbb{R}^{n+1} . Is *D* connected? Prove your answer. 3. Prove the following:

(a) If X is a compact space and $f:X\to Y$ is a continuous surjective map, then Y is compact.

(b) Let X be a compact space and Y be a Hausdorff space. Suppose that $f: X \to Y$ is a continuous bijection. Prove that f is a homeomorphism.

(c) Prove that the compactness assumption in (b) is necessary.

- 4. Let X, Y be Hausdorff topological spaces. Prove the followings:
 - (a) Prove that $X \times Y$ is compact if and only if X and Y are compact.

(b) Prove that $X\times Y$ is path–connected if and only if X and Y are path–connected.

Preliminary Exam DURATION: 3 hours

1- Prove or disprove. $(X, Y \text{ are topological spaces}, A, B \text{ are subsets of a topological space } X, \overline{A}$ denotes the closure of the set A, A' denotes the set of limit points of the set A, A° denotes the interior of the set $A, \partial A$ denotes the boundary of the set A.)

- (a) $(A \cup B)^{\circ} = A^{\circ} \cup B^{\circ}$.
- (b) $f^{-1}(C') = (f^{-1}(C))'$ for any continuous function $f: X \to Y$ and for all $C \subset Y$.
- (c) If $A^{\circ} \neq \emptyset$, then $\overline{A^{\circ}} = \overline{A}$.
- (d) If A and B are connected and $A \cap B \neq \emptyset$, then $A \cap B$ is connected.
- (e) If X is connected and if A is a proper subset of X (that is $A \neq \emptyset$ and $A \neq X$), then $\partial A \neq \emptyset$.

2- Let

$$\mathcal{T} = \{(-\infty, a) \, | \, a \in \mathbb{R}\}.$$

- (a) Show that \mathcal{T} is a topology on \mathbb{R} .
- (b) Compare this topology with the standard topology on \mathbb{R} .
- (c) Let A = (-1, 1) and $B = (-\infty, 1]$. Find the interiors A° , B° and the closures \overline{A} , \overline{B} of the sets A, B in this topology.

3- Let X, Y be topological spaces where Y is Hausdorff. Let $A \subset X$ be dense in X, i.e. $\overline{A} = X$. Let $f, g: X \to Y$ be continuous functions such that f(a) = g(a) for all $a \in A$. Show that f = g.

4- Let X, Y be topological spaces, $a \in X$ and $C \subset Y$ be compact in Y. Suppose there is an open set N in $X \times Y$ such that $\{a\} \times C \subset N$. Show that there is an open set $U \subset X$ and an open set $V \subset Y$ such that $a \in U, C \subset V$ and $U \times V \subset N$.

Topology TMS EXAM September 19, 2014

Duration: 3 hrs.

1. Let \mathbb{Q} denote the set of rational numbers considered as a subspace of \mathbb{R} , the space of real numbers.

- (a) Show that one-point subsets are not open in \mathbb{Q} .
- (b) Show that \mathbb{Q} is totally disconnected (i.e. the only connected subsets are one-point sets $\{q\}, q \in \mathbb{Q}$).
- (c) Prove that \mathbb{Q} is not locally connected.
- **2.** Let $\{X_{\alpha} | \alpha \in \Lambda\}$ be a family of spaces and let $A_{\alpha} \subset X_{\alpha}$ for each $\alpha \in \Lambda$.
 - (a) Show that if A_{α} is closed in X_{α} , then $\prod A_{\alpha}$ is closed in $\prod X_{\alpha}$. Why does this imply that $\overline{\prod A_{\alpha}} \subset \prod \overline{A_{\alpha}}$?
- (**b**) Show that $\prod \overline{A_{\alpha}} \subset \overline{\prod A_{\alpha}}$

3.

- (a) State Urysohn Lemma.
- (b) Show that a connected normal space having more than one point is uncountable.
- **4.** Let X be a compact Hausdorff space and $A \subset X$ be a closed subset.
 - (a) Show that $X \setminus A$ is a locally compact Hausdorff space.
 - (b) Show that the one-point compactification $(X \setminus A)^*$ of the space $X \setminus A$ is homeomorphic to the quotient space X/A. (Recall that X/A is the quotient space X/\sim where the equivalence classes are $\{A\}$ and the single point sets.)

TOPOLOGY TMS EXAM October 02 2015

Duration: 3 hours.

(1) Let $f: X \to Y$ be a map between topological spaces. Show that the following are equivalent.

- (i) f is continuous and open.
- (ii) $f^{-1}(\operatorname{Int}(B)) = \operatorname{Int}(f^{-1}(B))$ for all $B \subset Y$.
- (iii) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for all $B \subset Y$.

(2) Let (x_n) be a sequence of points of the space $\prod_{\alpha \in J} X_\alpha$ endowed with the product topology. Let $\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the projection mapping associated with the index β .

Show that the sequence (x_n) converges to x if and only if the sequence $\pi_\beta(x_n)$ converges to $\pi_\beta(x)$ for each $\beta \in J$.

Is this true if you use the box topology instead of the product topology?

(3) Let X be the subspace of \mathbb{R}^2 defined by $X = \bigcup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$ (X is the union of the circles with center (1/n, 0) and radius 1/n for n = 1, 2, 3, ...) Let Y be the quotient space formed by starting with \mathbb{R} and defining $x \sim y$ if either x = y or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic.

(4) Let X be metrizable. Show that the following are equivalent.

- (i) X is bounded under every metric that gives the topology of X.
- (ii) Every continuous function $\phi \colon X \to \mathbb{R}$ is bounded.
- (iii) X is limit point compact.

TOPOLOGY TMS EXAM October 02 2015

Duration: 3 hours.

(1) Let $f: X \to Y$ be a map between topological spaces. Show that the following are equivalent.

- (i) f is continuous and open.
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(3) Let X be the subspace of \mathbb{R}^2 defined by $X = \bigcup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$ (X is the union of the circles with center (1/n, 0) and radius 1/n for n = 1, 2, 3, ...) Let Y be the quotient space formed by starting with \mathbb{R} and defining $x \sim y$ if either x = y or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic.

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- (i) X is bounded under every metric that gives the topology of X.
- (ii) Every continuous function $\phi \colon X \to \mathbb{R}$ is bounded.
- (iii) X is limit point compact.

METU MATHEMATICS DEPARTMENT TMS EXAM IN TOPOLOGY

FEBRUARY 18, 2016

1. Parts are unrelated!

3.

a) Construct a topology τ on the interval [0,1) so that it becomes homeomorphic to the unit circle

$$S^{1} = \{(x, y) \in \mathbb{R}^{2} \mid x^{2} + y^{2} = 1\}$$

with the topology inherited from the standard topology of \mathbb{R}^2 . Find a homeomorphism $f:[0,1) \to S^1$, where [0,1) has the topology τ .

 $(\phi : [0,1] \rightarrow [0,1)$ is discontinuous at infinitely many points, where both interval are equipped with the standard topology inherited from the real line. c) Find a homeomorphism $\psi: [0, \infty) \to (0, 1]$.

2. A continuous map between two topological spaces is called proper if the preimage of any compact set is compact. Parts are unrelated!

- **S** a) Is there a proper map $f : \mathbb{R} \to [0, 1]$, where [0, 1] has its standard topology? Prove your answer.
- **b**) Show that $g: \mathbb{R} \to \mathbb{R}, g(x) = x^2$, is proper.
- $(\mathbf{Q} \mathbf{c})$ Let $h : X \to Y$ be a continuous map of Hausdorff topological spaces, where X is compact. Show that h is proper.
- $l \bigcirc a$) Define an equivalence relation on \mathbb{R}^2 (equipped with the standard topology) as follows:

 $(x_0, y_0) \sim_1 (x_1, y_1) \iff x_0^2 + y_0 = x_1^2 + y_1$.

Show that the quotient space \mathbb{R}^2/\sim_1 is homeomorphic to the real line with its standard topology.

b) Instead of the above equivalence relation \sim_1 in Part (a) consider the relation \sim_2 defined 15

 $(x_0, y_0) \sim_2 (x_1, y_1) \iff x_0^2 + y_0^2 = x_1^2 + y_1^2$.

Is the quotient space \mathbb{R}^2/\sim_2 homeomorphic to \mathbb{R}^2/\sim_1 ? Prove your answer!

4. Let (X, d) be a metric space. A function $f: X \to X$ is called an isometry if d(x, y) =d(f(x), f(y)), for all $x, y \in X$.

- (X) a) Prove that if (X,d) is compact and connected then any isometry $f: X \to X$ is a homeomorphism homeomorphism.
- $(\mathbf{q}$ b) Prove that any isometry $f: \mathbb{R} \to \mathbb{R}$ is a homeomorphism, where the real line has its standard absolute value metric.
- \leq c) Find an isometry $f: [0, \infty) \to [0, \infty)$, which is not a homeomorphism.

Topology TMS EXAM February 16, 2017

Duration: 3 hr.

1. a) Show that every subspace of a second countable space is Lindelöf.

b) Show that any base for the open sets in a second countable space has a countable subfamily which is a base.

2. Show that continuous Hausdorff image of a compact locally connected space is compact and locally connected.

3. a) Define a (topological) imbedding.

b) Let X be a compact space, Y be a Hausdorff space and $g: X \to Y$ be continuous one-to-one map. Show that g is a topological imbedding.

c) Let X and Y be two arbitrary spaces and $f: X \to Y$ be a continuous map. Show that the map $F: X \to X \times Y$ which is given by F(x) = (x, f(x)) is a topological imbedding.

4. Let X be a set and $\{(X_{\alpha}, \tau_{\alpha})\}|\alpha \in \Lambda$ be a collection of spaces. For each α , let $f_{\alpha} : X \to X_{\alpha}$ be a map. Recall that the *weak topology* on X induced by the collection $\{f_{\alpha}|\alpha \in \Lambda\}$ is the topology τ on X for which the sets $f_{\alpha}^{-1}(U_{\alpha})$ for $\alpha \in \Lambda$ and U_{α} is open in X_{α} , form a subbase.

a) Show that τ is the smallest topology on X making each f_{α} continuous.

b) Let $\{\tau_{\alpha} | \alpha \in \Lambda\}$ be a family of topologies on a fixed set X and denote by X_{α} the space consisting of the set X with the topology τ_{α} . Denote the identity function from X to the space X_{α} by i_{α} . Let τ denote the weak topology induced on X by the collection $\{i_{\alpha} | \alpha \in \Lambda\}$. Exhibit a homeomorphism F from X to the diagonal Δ in the product space $\prod X_{\alpha}$ and verify that it is indeced a homeomorphism. (Note: $\Delta = \{x \in \prod X_{\alpha} | x_{\alpha} = x_{\beta} \text{ for all } \alpha, \beta\}$.)